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PIERRE AUGER OBSERVATORY

HIRSAP Helmholtz International Research School for Astroparticle Physics

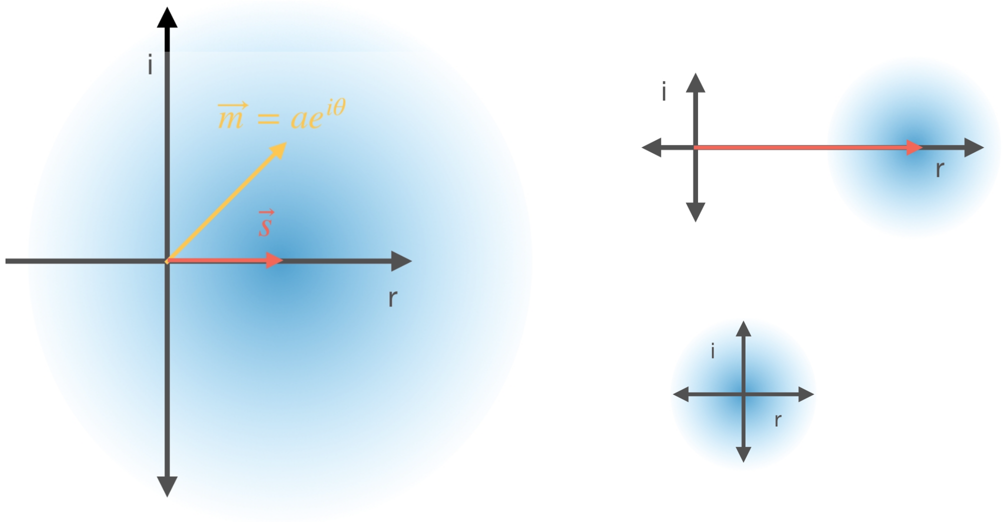
KIT Karlsruher Institut für Technologie

UNSAM

ARENA 2024 - Chicago 11th June 2024

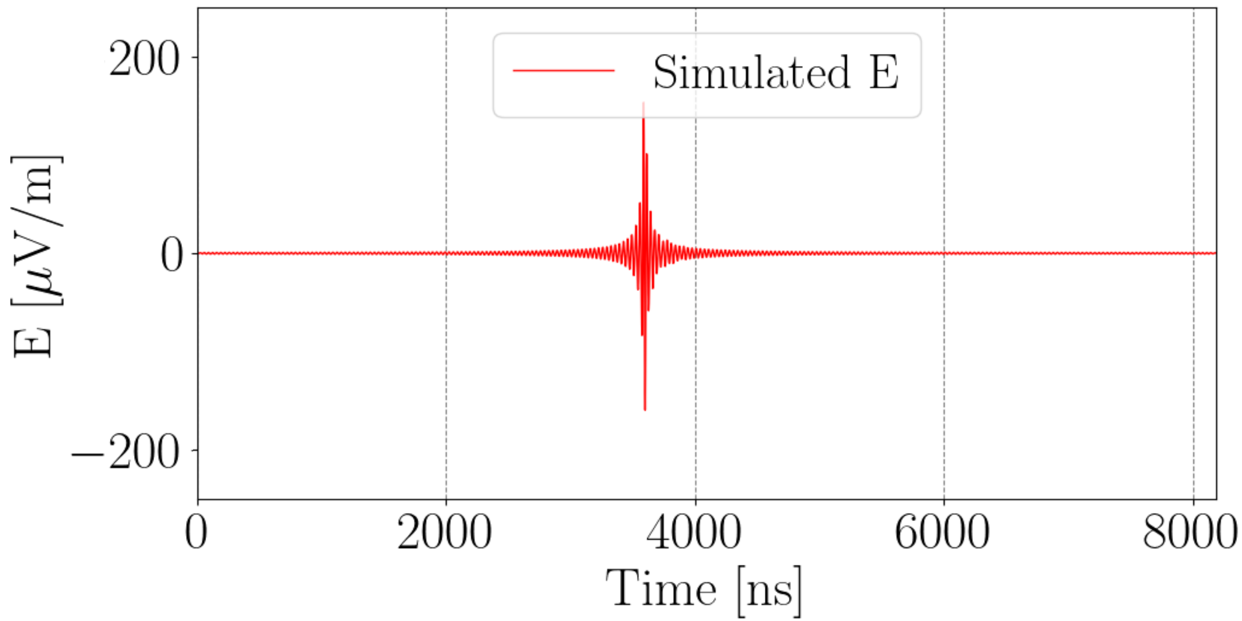
Quantifying energy fluences and their uncertainties in the presence of noise

Sara Martinelli, Dr. Tim Huege, Dr. Diego Ravignani, Dr. Harm Schoorlemmer



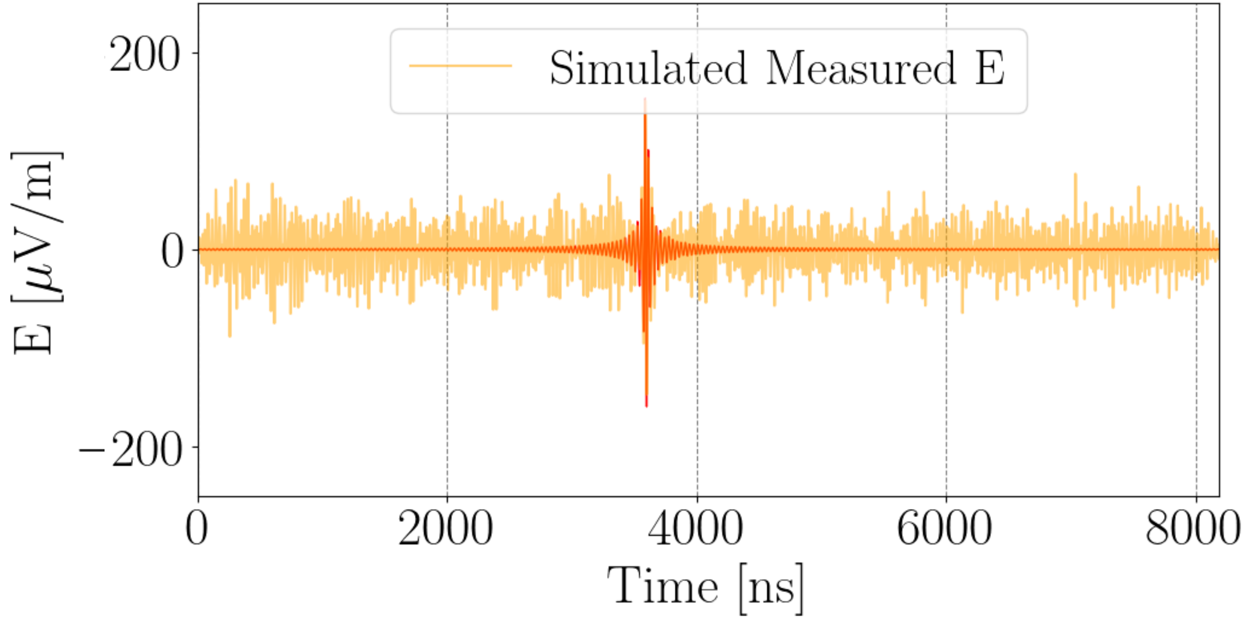
It is the energy deposit per unit area in terms of radio waves. The total fluence at a given antenna position is the time integral of the Poynting vector:

$$f_{\text{tot}}(\vec{r}) = \epsilon_0 \, c \, \Delta t \sum_{\text{pol}}^{3} \left(\sum_{j} E_{\text{pol}}^2(\vec{r}, t_j) \right) = \sum_{\text{pol}}^{3} f_{\text{pol}}(\vec{r})$$



We need a method to estimate the energy fluence in the presence of noise.

The noise subtraction method is largely used within the radio community:
- works well for large signal-to-noise ratio (SNR)



SNR_{pol}=60

We need a method to estimate the energy fluence in the presence of noise.

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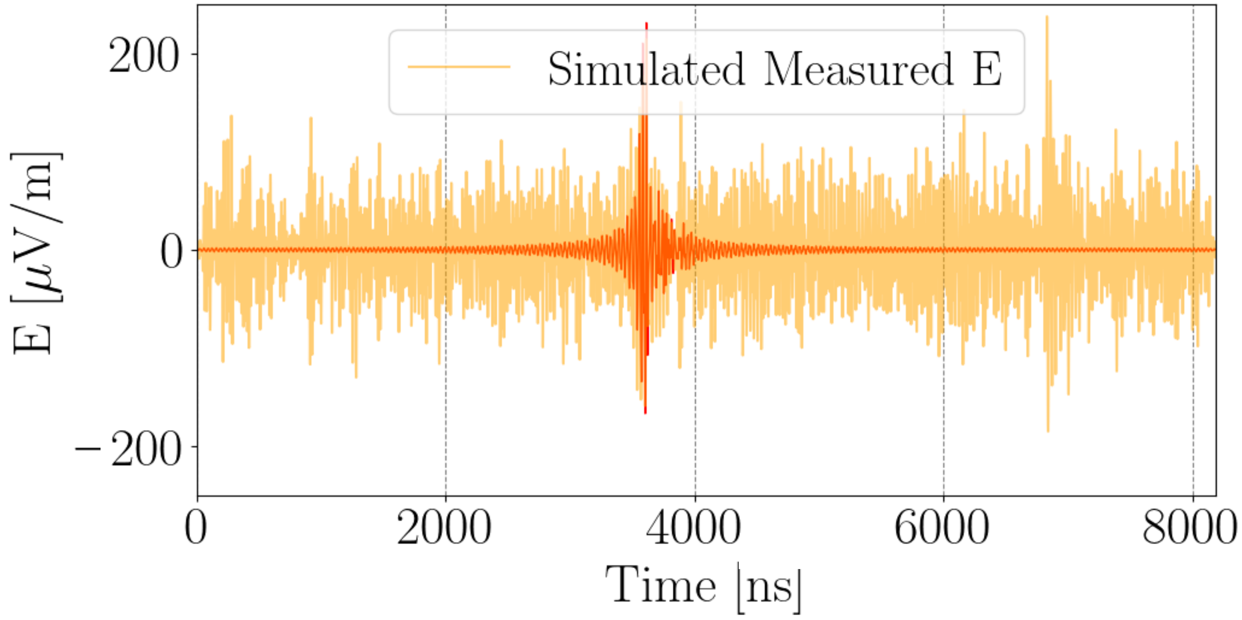

SNR_{pol}=4

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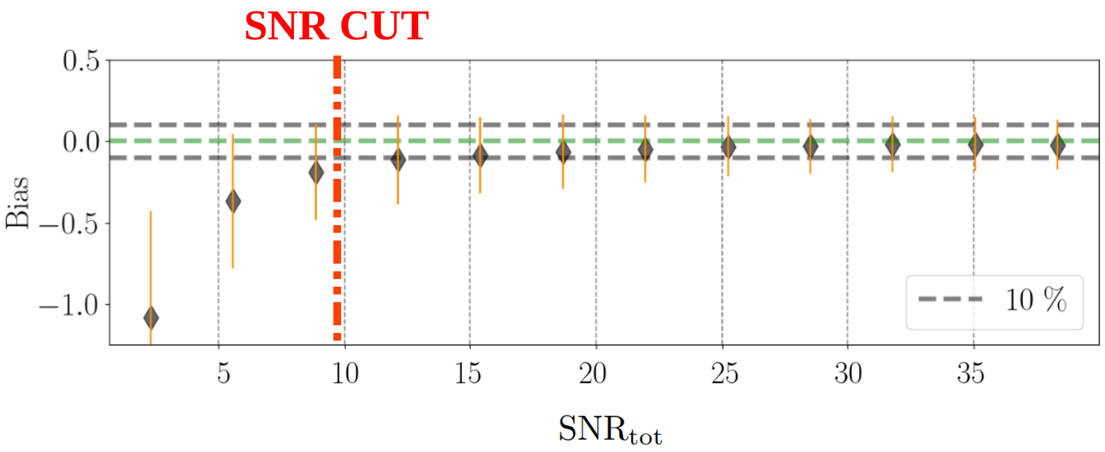
We need a method to estimate the energy fluence in the presence of noise.

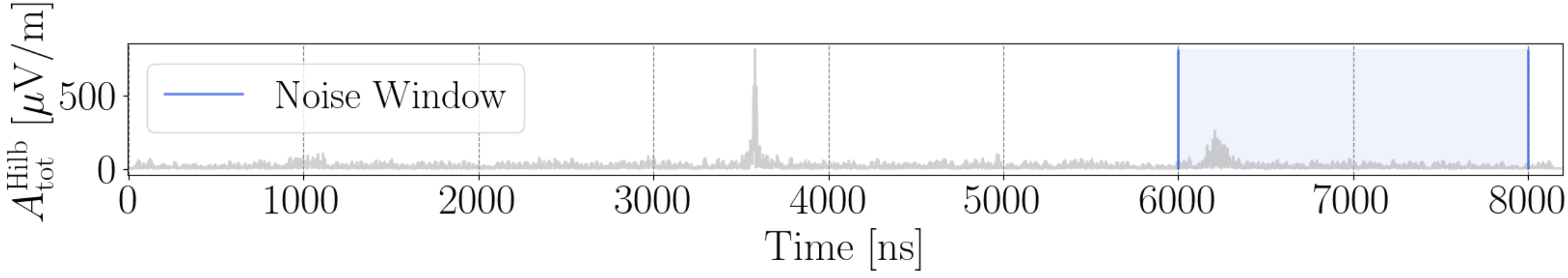
The noise subtraction method is largely used within the radio community:
- works well for large signal-to-noise ratio (SNR)
- an SNR threshold cut is usually imposed



(as defined in the next slide)

SNR Definition

$$A_{\mathrm{tot}}^{\mathrm{Hilb}}(\vec{r},t) = \sqrt{\sum_{\mathrm{pol}}^{3} \left| E_{\mathrm{pol}}^{\mathrm{Hilb}}(\vec{r},t) \right|^2}$$

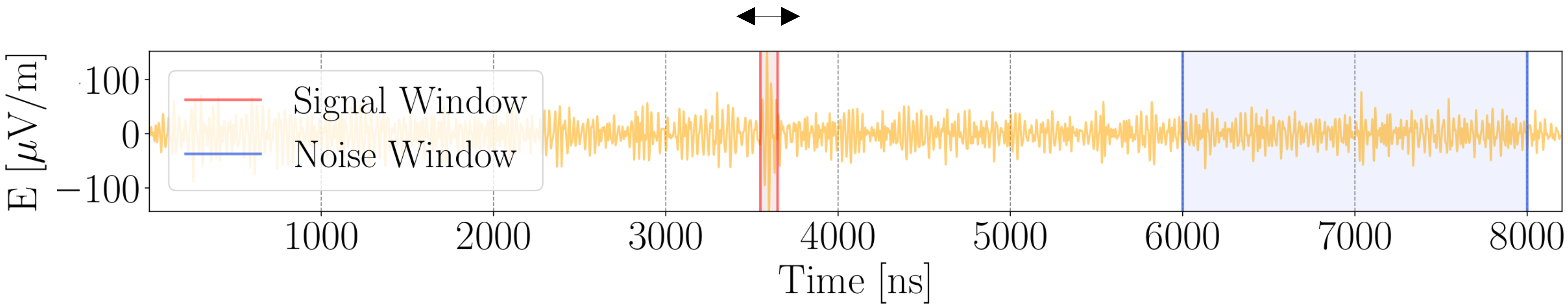


$$\mathrm{SNR}_{\mathrm{tot}} = \left(\frac{A_{\mathrm{tot}}^{\mathrm{Hilb}}\big|_{\mathrm{max}}}{A_{\mathrm{tot}}^{\mathrm{RMS}}} \right)^2$$

(similar definition at polarisation level)

pol. — Definition of the signal window

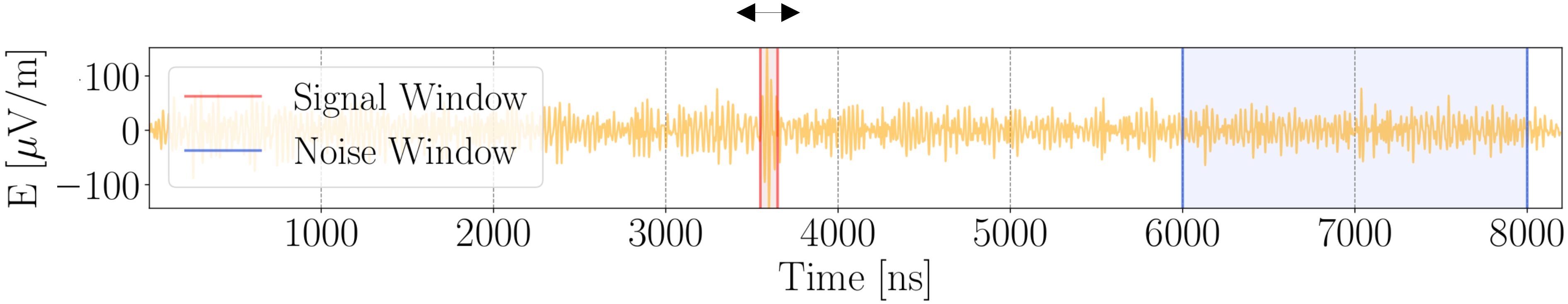
Algorithm to find the maximum $A_{\mathrm{tot}}^{\mathrm{Hilb}}|_{\mathrm{max}}$



Noise Subtraction Method

pol. — Definition of the signal window

$$\hat{f}_{\mathrm{pol}}(\vec{r}) = \epsilon_0\, c\, \Delta t \left(\sum_{t_j=t_1}^{t_2} E^2_{\mathrm{pol}}(\vec{r}, t_j) + \ ... \right)$$



pol.

Definition of the signal window

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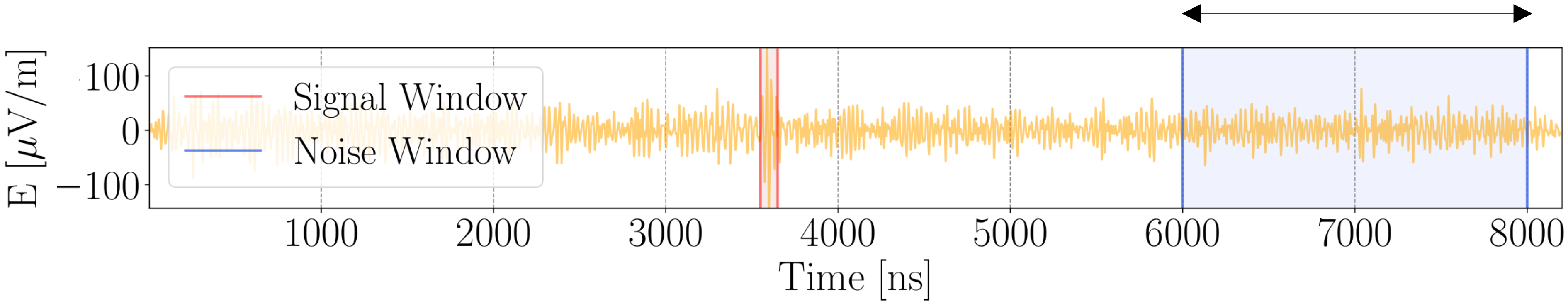
Definition of the noise window



pol.

Definition of the signal window

$$\hat{f}_{\text{pol}}(\vec{r}) = \epsilon_0\, c\, \Delta t \left(\sum_{t_j=t_1}^{t_2} E^2_{\text{pol}}(\vec{r}, t_j) + \ \ldots \right)$$

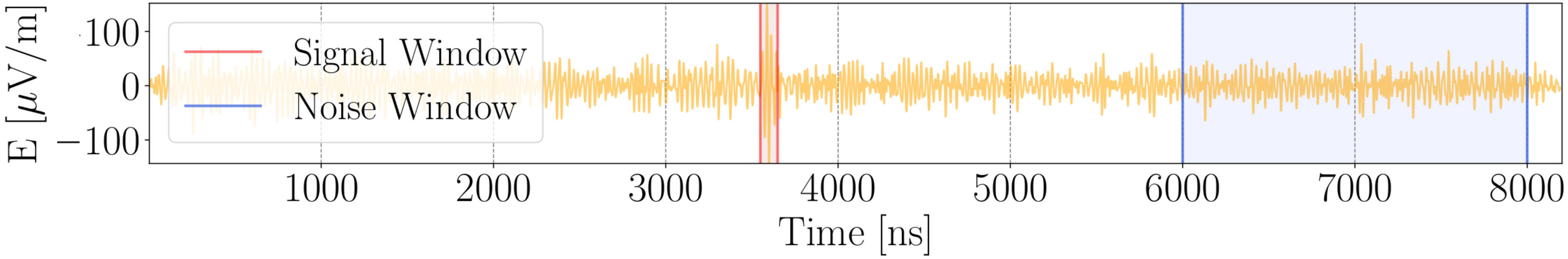
Definition of the noise window

Fluence estimator & uncertainty

$$\hat{f}_{\text{pol}}(\vec{r}) = \epsilon_0\, c\, \Delta t \left(\sum_{t_j=t_1}^{t_2} E^2_{\text{pol}}(\vec{r}, t_j) - \frac{t_2 - t_1}{t_4 - t_3} \sum_{t_j=t_3}^{t_4} E^2_{\text{pol}}(\vec{r}, t_j) \right)$$

Subtraction of the normalized noise fluence

→ the estimator can be negative



pol.

Definition of the signal window

$$\hat{f}_{\mathrm{pol}}(\vec{r}) = \epsilon_0\, c\, \Delta t \left(\sum_{t_j=t_1}^{t_2} E^2_{\mathrm{pol}}(\vec{r}, t_j) + \; ... \right)$$

Definition of the noise window

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Underestimated

(backup)

$$\delta\big(\hat{f}_{\mathrm{pol}}(\vec{r})\big) = \sqrt{4\, \epsilon_0\, c\, \Delta t\, \hat{f}_{\mathrm{pol}}(\vec{r})\, \sigma_e^2 \; + \; 2\, (\epsilon_0\, c)^2\, \Delta t\, \sigma_e^4}$$

It assumes the measured amplitude is the sum of the pulse and the noise (Gaussian- distributed)

pol.

Definition of the signal window

$$\hat{f}_{\text{pol}}(\vec{r}) = \epsilon_0\, c\, \Delta t \left(\sum_{t_j=t_1}^{t_2} E^2_{\text{pol}}(\vec{r}, t_j) + \ \dots \right)$$

Definition of the noise window

Fluence estimator & uncertainty

$$\hat{f}_{\text{pol}}(\vec{r}) = \epsilon_0\, c\, \Delta t \left(\sum_{t_j=t_1}^{t_2} E^2_{\text{pol}}(\vec{r}, t_j) - \frac{t_2 - t_1}{t_4 - t_3} \sum_{t_j=t_3}^{t_4} E^2_{\text{pol}}(\vec{r}, t_j) \right)$$

$$\delta\big(\hat{f}_{\text{pol}}(\vec{r})\big) = \sqrt{4\, \epsilon_0\, c\, \Delta t\, \hat{f}_{\text{pol}}(\vec{r})\, \sigma_e^2 \ + 2\, (\epsilon_0\, c)^2\, \Delta t\, \sigma_e^4}$$

$$\hat{f}_{\text{tot}}(\vec{r}) = \sum_{\text{pol}}^{3} \hat{f}_{\text{pol}}(\vec{r})$$

Error propagation

pol.

Definition of the signal window

Definition of the noise window

Fluence estimator & uncertainty

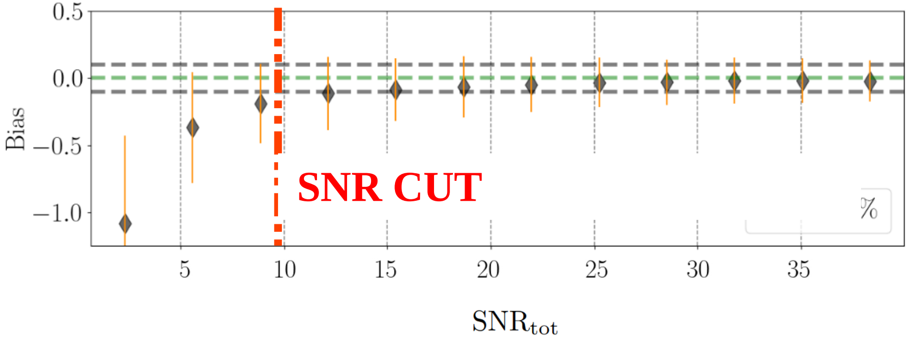
$$\hat{f}_{\mathrm{tot}}(\vec{r}) = \sum_{\mathrm{pol}}^{3} \hat{f}_{\mathrm{pol}}(\vec{r})$$

Error propagation

In summary:

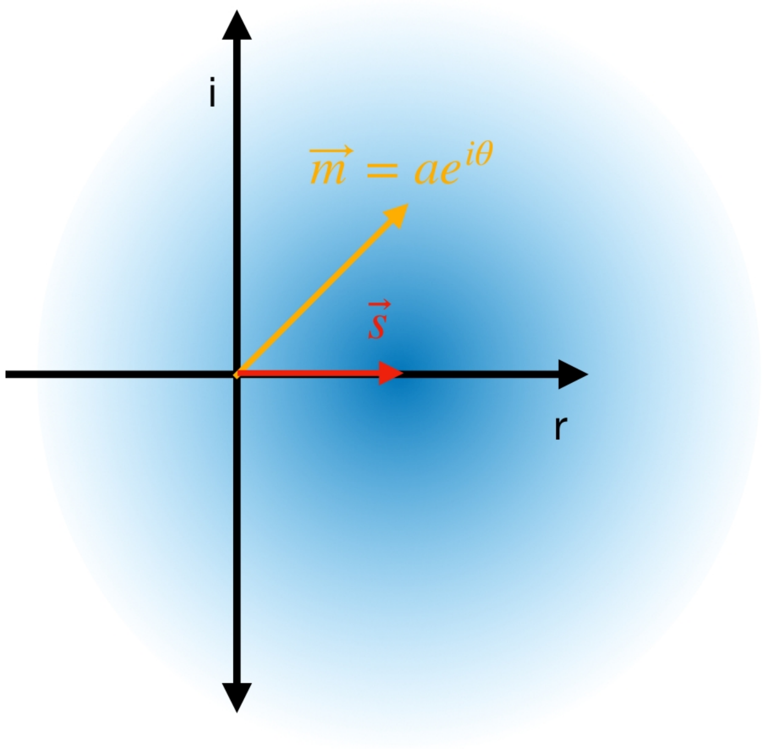
- Biased at low SNR values (SNR cut)
- Uncertainties underestimated

Why? Can we do better?



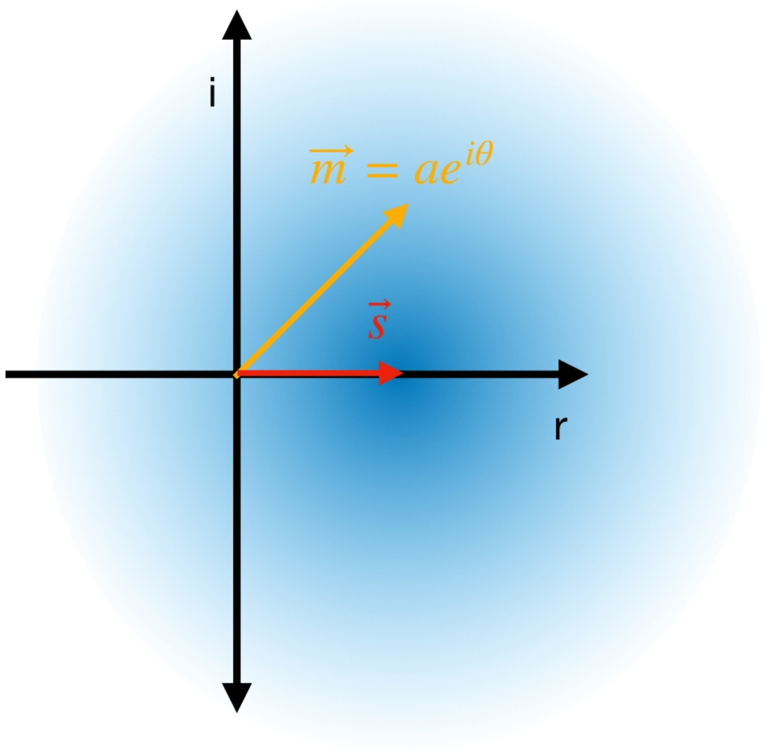
$\mathrm{SNR}_{\mathrm{tot}} = \left(\frac{A^{\mathrm{Hilb}}_{\mathrm{tot}}|_{\max}}{A^{\mathrm{RMS}}_{\mathrm{tot}}}\right)^2$

Radio measurements have both an amplitude and a phase



- The signal and the **random noise** can add up constructively or destructively.

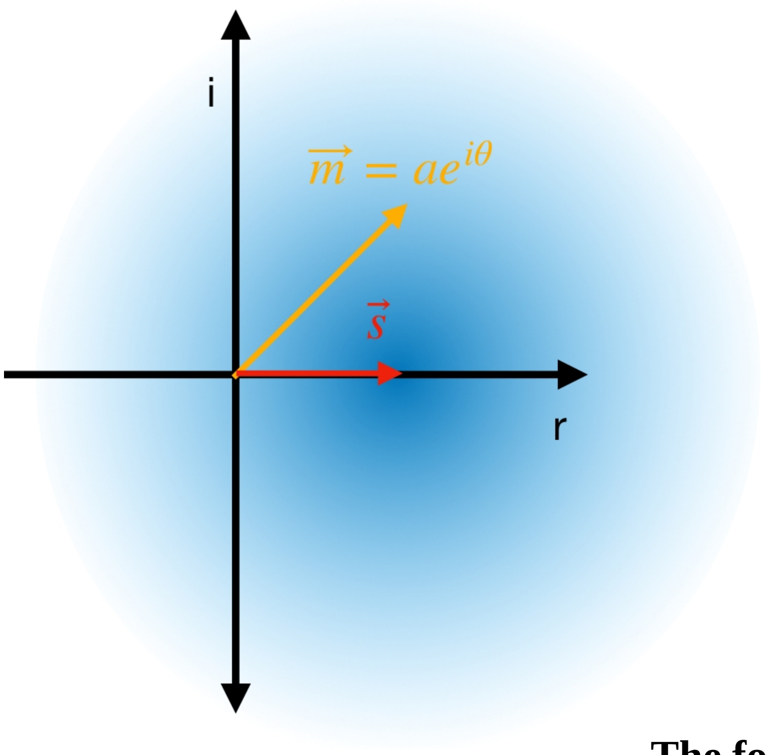
Reference: Chapter 2.9 from J. W. Goodman, Statistical Optics (2015)



Radio measurements have both an amplitude and a phase

- The signal and the **random noise** can add up constructively or destructively.
- Our measurement can be expressed as the sum of constant known phasor s and a random phasor sum (Rayleigh-distributed noise).

Reference: Chapter 2.9 from J. W. Goodman, Statistical Optics (2015)



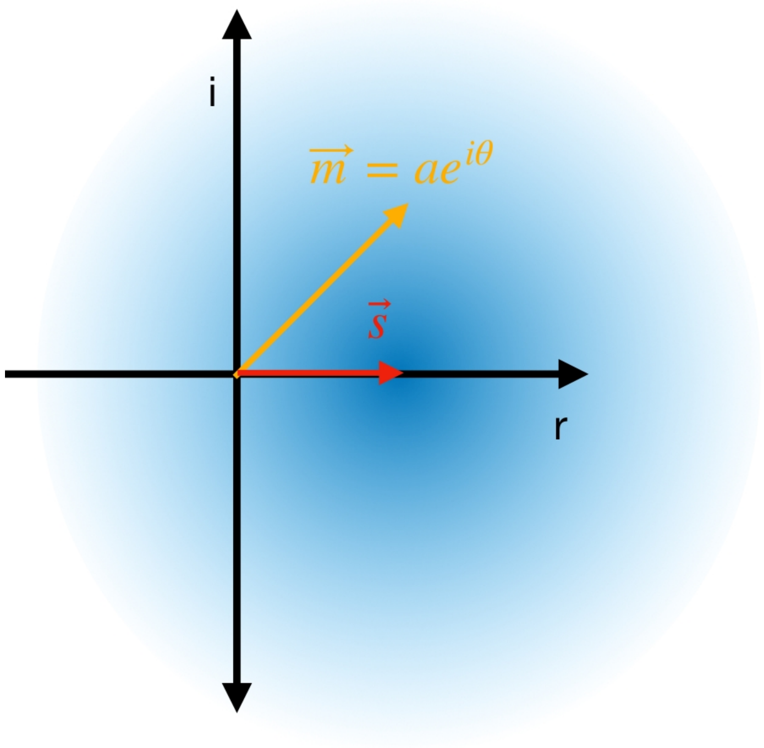
Radio measurements have both an amplitude and a phase

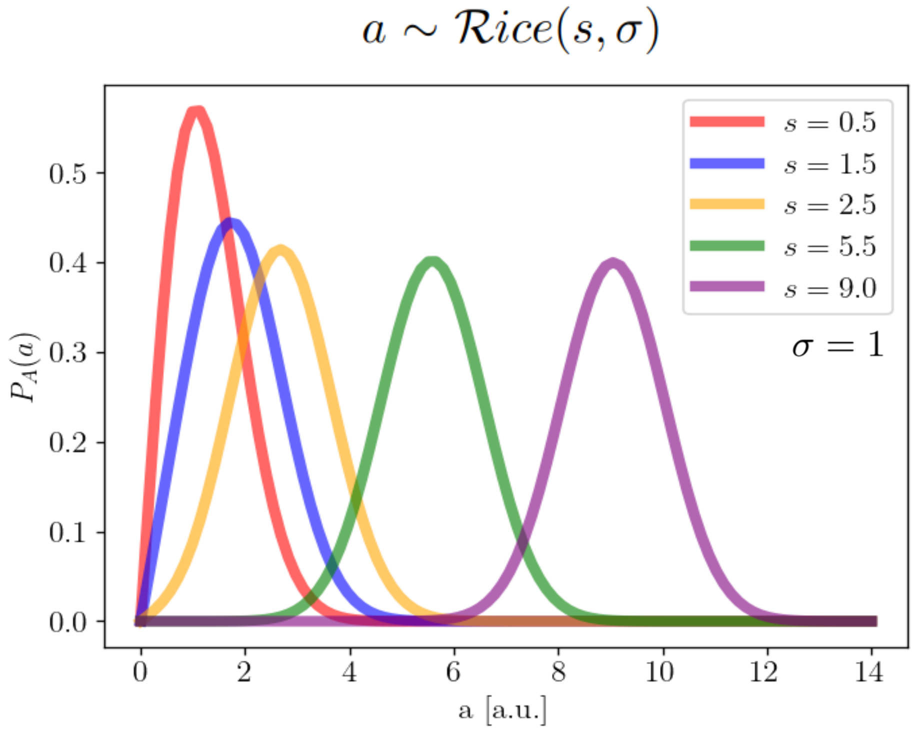
- The signal and the **random noise** can add up constructively or destructively.
- Our measurement can be expressed as the sum of constant known phasor s and a random phasor sum (Rayleigh-distributed noise).
- The **marginal P.D.F**. of the measured **amplitude** is the Rice distribution.

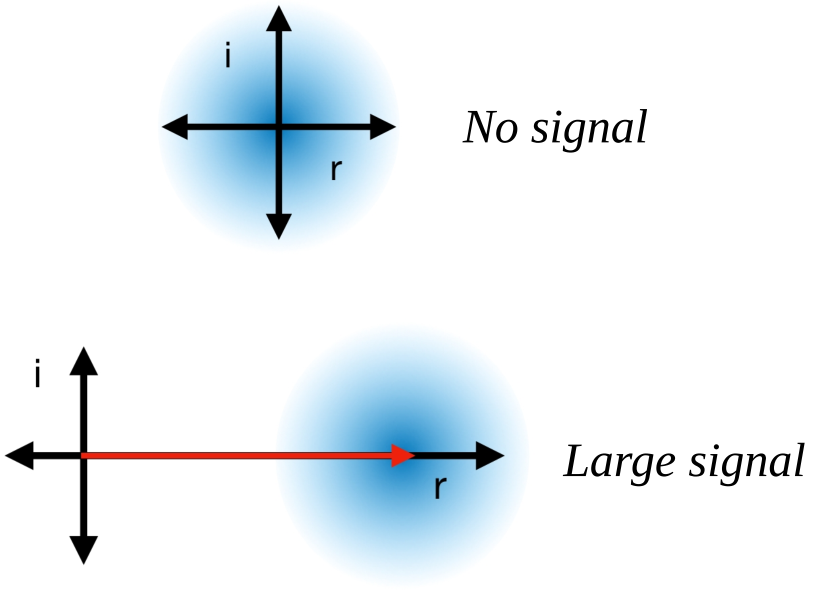
$$p_A(a|s,\sigma) = \begin{cases} \frac{a}{\sigma^2} \cdot \exp\left(-\frac{a^2+s^2}{2\sigma^2}\right) \cdot I_0\left(\frac{as}{\sigma^2}\right) & \text{a} > 0 \\ 0 & \text{otherwise} \end{cases}$$

The formalism is valid for both time and frequency domains

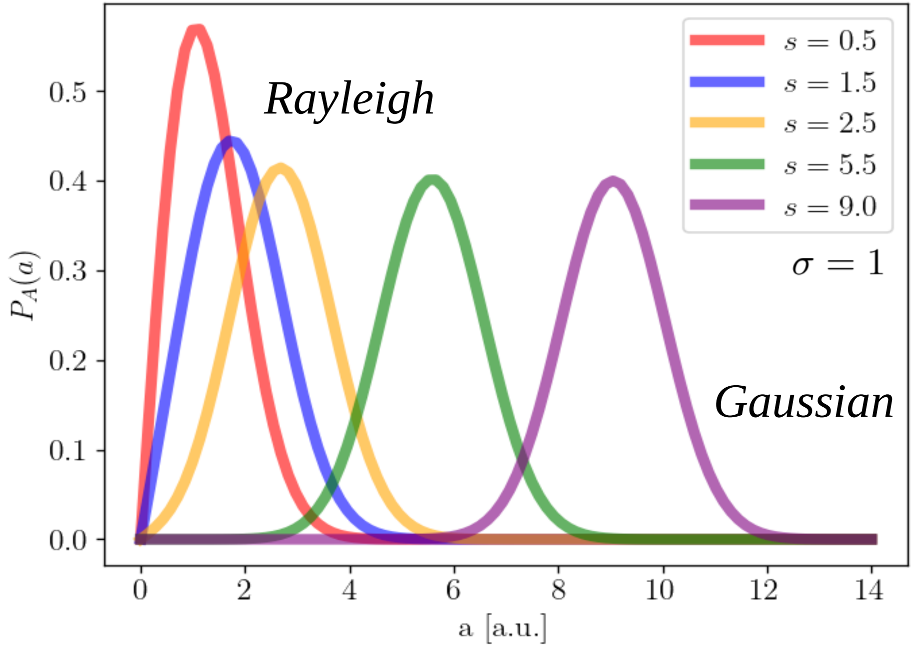
Reference: Chapter 2.9 from J. W. Goodman, Statistical Optics (2015)


i
$\vec{m} = ae^{i\theta}$
$\vec{s}$
r


$a \sim \mathcal{R}ice(s, \sigma)$
$s = 0.5$
$s = 1.5$
$s = 2.5$
$s = 5.5$
$s = 9.0$
$\sigma = 1$
$P_A(a)$
a [a.u.]
0.0
0.1
0.2
0.3
0.4
0.5
0
2
4
6
8
10
12
14



$$a \sim \mathcal{R}ice(s, \sigma)$$



pol.

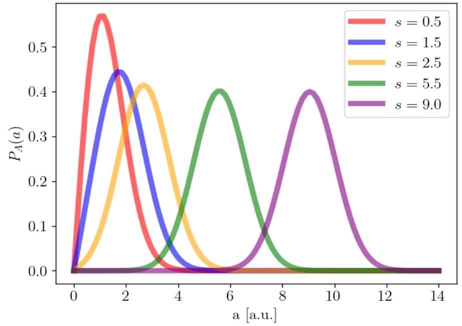
Rice-distribution Based Method

$$\hat{f}_{\text{tot}}(\vec{r}) = \sum_{\text{pol}}^{3} \hat{f}_{\text{pol}}(\vec{r})$$

Error propagation

We developed a method:

- using the statistical background based on the Rice distribution to build a fluence estimator



- evaluating the fluence in the frequency domain:

$$f_{\text{pol}}(\vec{r}) = \epsilon_0\, c\, \Delta t \sum_{j=0}^{N-1} E^2_{\text{pol}}(\vec{r}, t_j) = 2\, \epsilon_0\, c\, \frac{\Delta t}{N} \sum_{j=0}^{M-1} |D_{\text{pol}}(v_j)|^2$$

(Parseval's Theorem)

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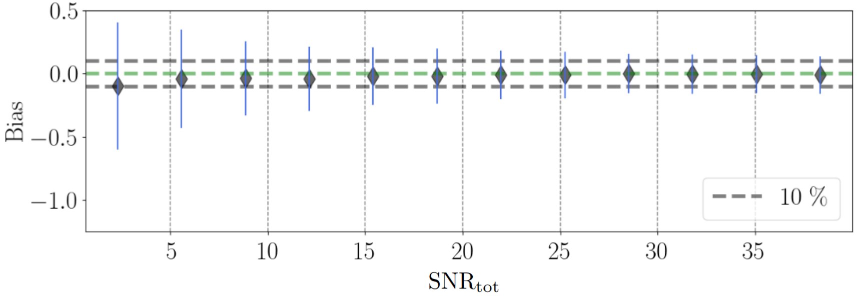
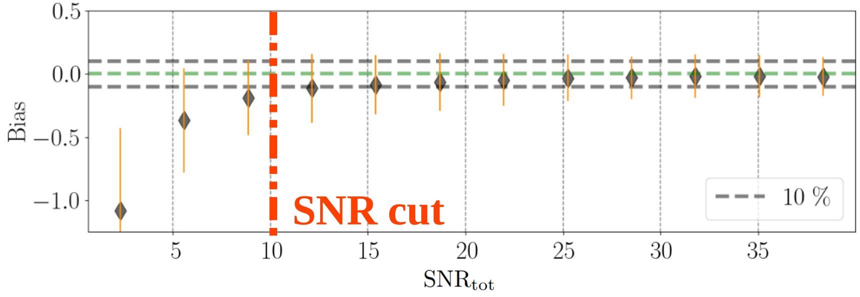
pol.

Rice-distribution Based Method



pol.

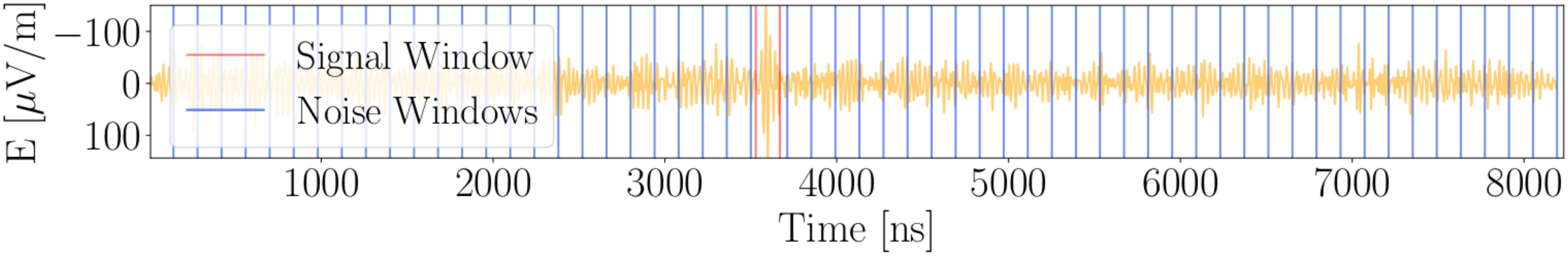
Noise Subtraction Method



Rice-distribution Based Method

pol. — Definition of the signal and noise windows

Tukey window function

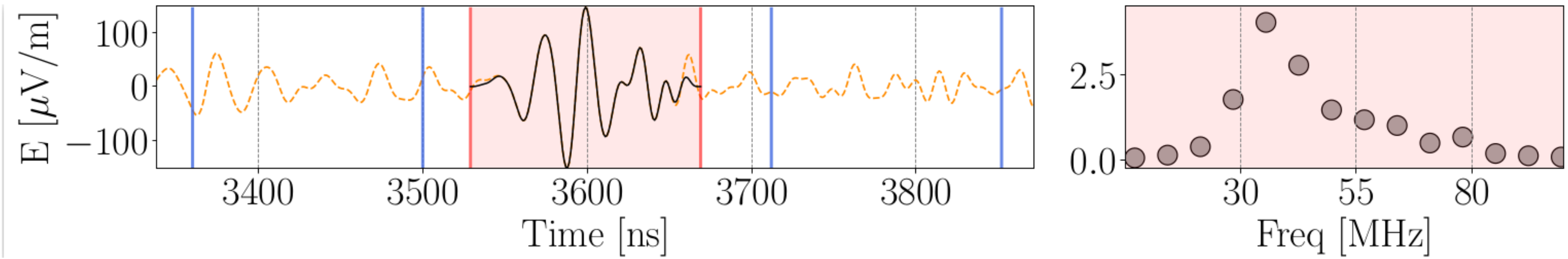


pol.

Definition of the signal and noise windows

Signal window: Tukey function and FFT

Tukey window function



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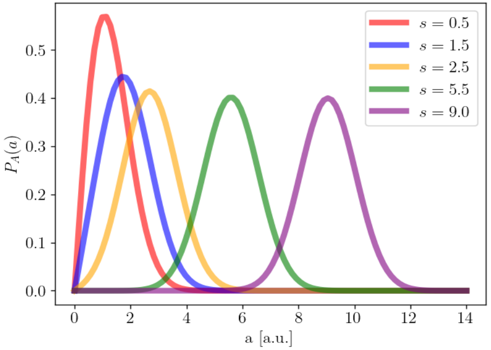
pol.

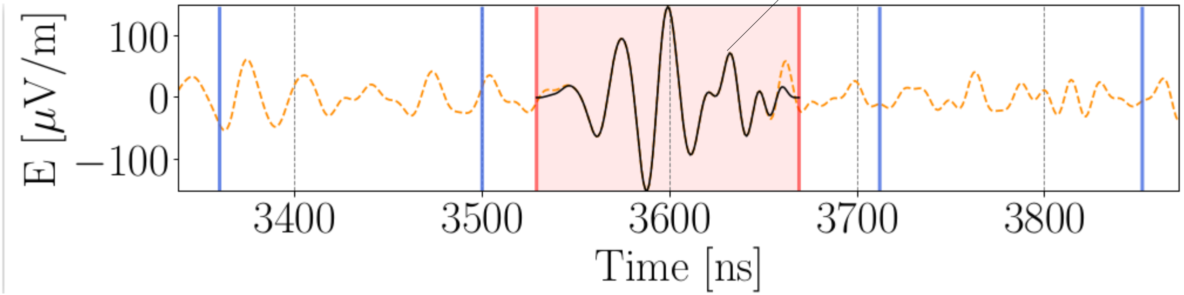
Definition of the signal and noise windows

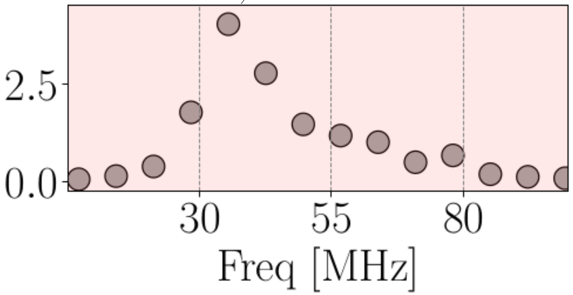
Signal window: Tukey function and FFT



$a \sim \mathcal{R}ice(s, \sigma)$

$a \sim \mathcal{R}ice(s, \sigma)$





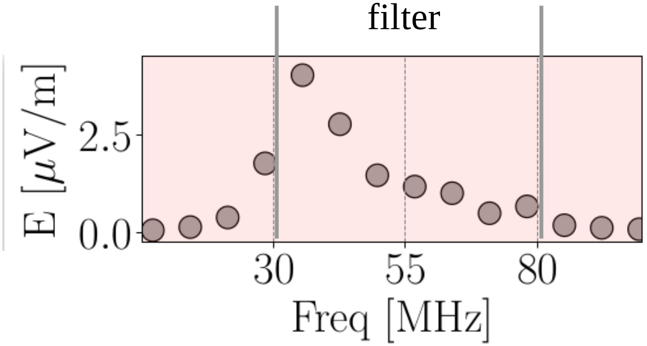
pol.

Definition of the signal and noise windows

Signal window: Tukey function and FFT

M frequencies

$$\hat{f}_{\mathrm{a}} = K \sum_{j=0}^{M-1} a^2(v_j) = \sum_{j=0}^{M-1} \hat{f}_{\mathrm{a}}(\nu_j)$$ measured



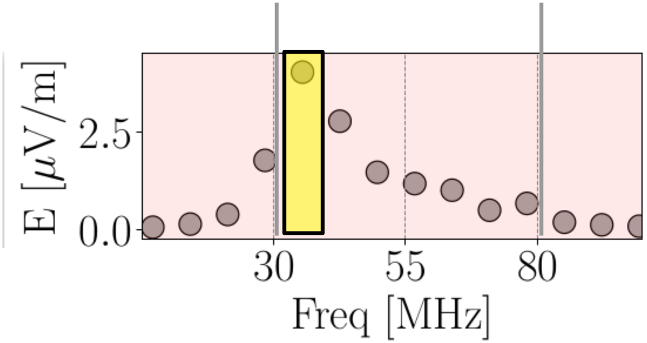
pol.

Definition of the signal and noise windows

Signal window: Tukey function and FFT

j-th frequency

$\hat{f}_{\mathrm{a}} = a^2$ measured



pol.

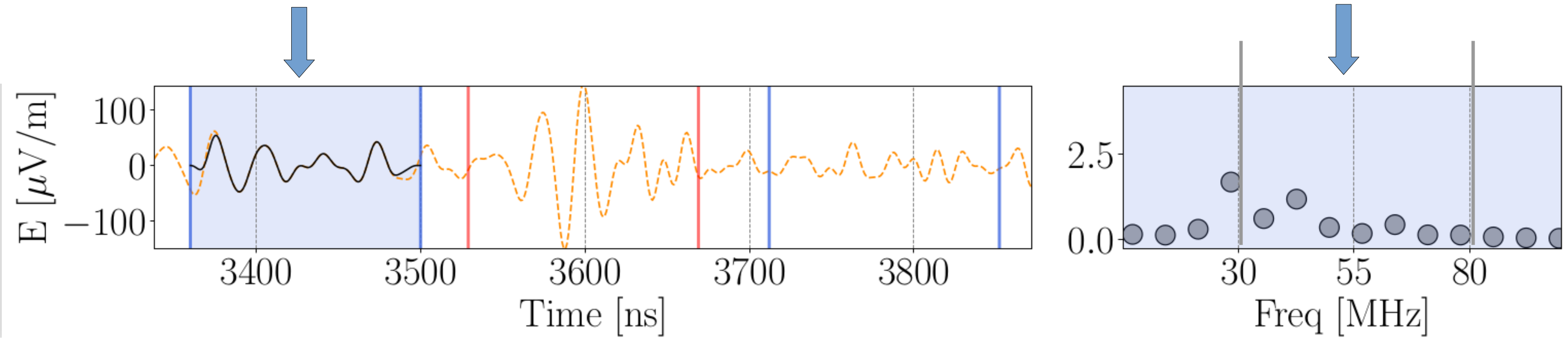
Definition of the signal and noise windows

Signal window: Tukey function and FFT

Noise windows: Tukey function, FFT

j-th frequency

$$\hat{f}_a = a^2$$



pol.

Definition of the signal and noise windows

Signal window: Tukey function and FFT

Noise windows: Tukey function and FFT

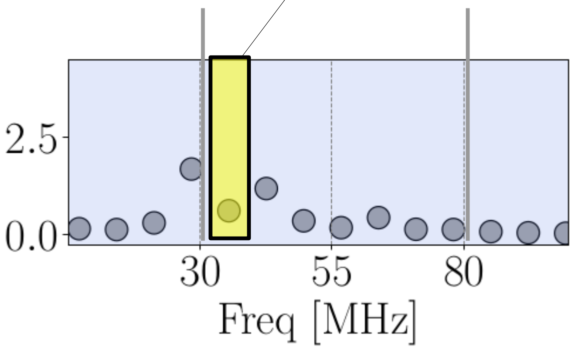
j-th frequency

$$\hat{f}_{\mathrm{a}} = a^2$$

$$\hat{f}_{\mathrm{n}} = \frac{1}{N}\sum_{i=0}^{N-1} n_i^2$$

$$n_i \sim \ Rayleigh(\sigma)$$

N times



pol.

Definition of the signal and noise windows

Signal window: Tukey function and FFT

Noise windows: Tukey function and FFT

Estimator and uncertainty

j-th frequency

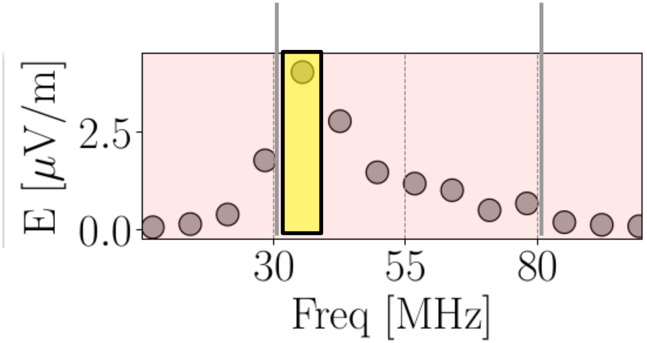
$$\hat{f}_{\mathrm{a}} = a^2$$

$$\hat{f}_{\mathrm{n}} = \frac{1}{N}\sum_{i=0}^{N-1} n_i^2$$

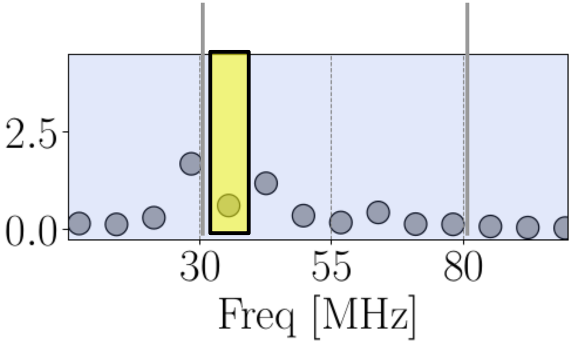
$$\hat{f}_s = \begin{cases} \hat{f}_a - \hat{f}_n \text{ if } \hat{f}_a \geq \hat{f}_n \\ 0 \text{ if } \hat{f}_a < \hat{f}_n \end{cases}$$

$$\delta f_s = \sqrt{\hat{f}_n\,(\hat{f}_n + 4\,\hat{f}_s)}$$

(Derivation in the backup)



N times



pol.

Definition of the signal and noise windows

Signal window: Tukey function and FFT

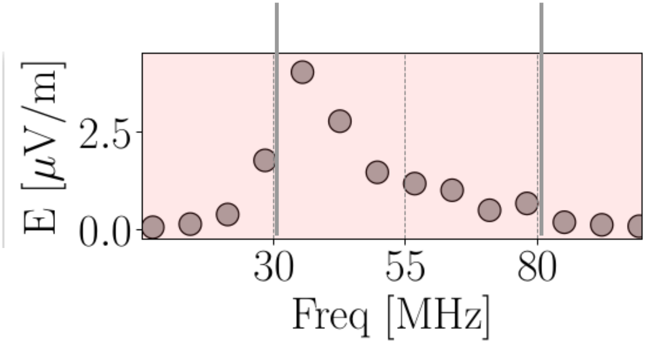
Noise windows: Tukey function and FFT

Estimator and uncertainty **M times!**

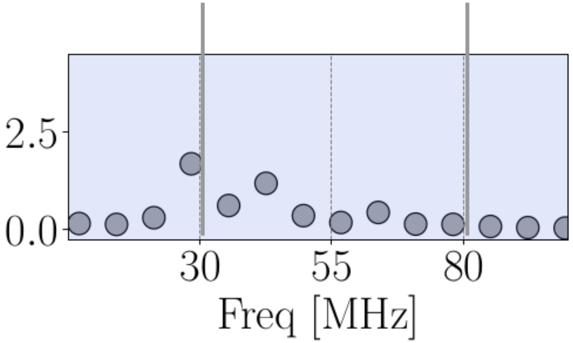
M frequencies!

$$\hat{f}_\mathrm{s} = \sum_{j=0}^{M-1} \hat{f}_\mathrm{s}(\nu_j)$$

$$Var(\hat{f}_\mathrm{s}) = \sum_{j=0}^{M-1} Var\big(\hat{f}_\mathrm{s}(\nu_j)\big)$$



N times



pol.

Definition of the signal and noise windows

Signal window: Tukey function and FFT

Noise windows: Tukey function and FFT

Estimator and uncertainty

$$\hat{f}_{\text{tot}}(\vec{r}) = \sum_{\text{pol}}^{3} \hat{f}_{\text{pol}}(\vec{r})$$

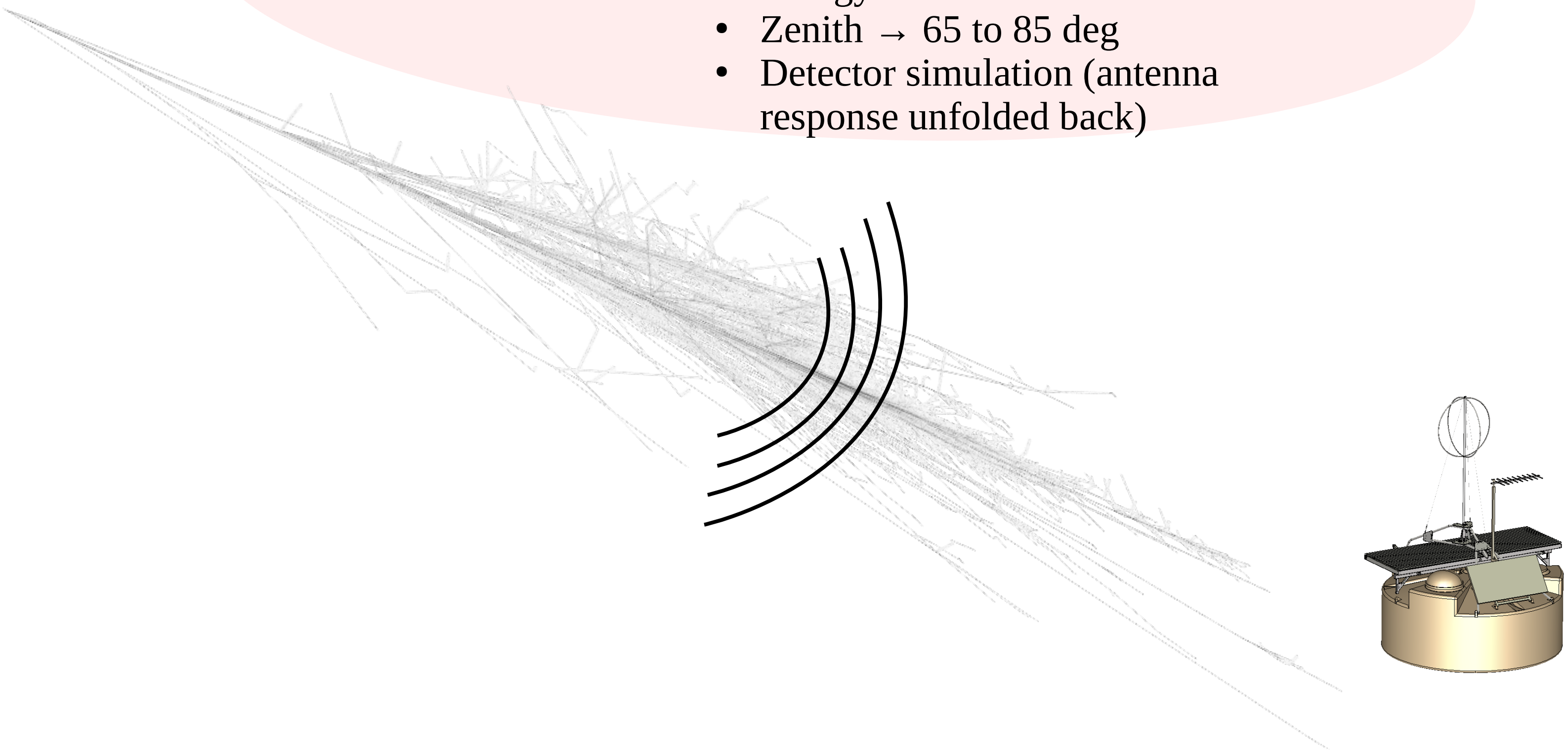
Error propagation

We can now compare the noise subtraction method and the Rice method...

Simulated Data Set

Simulations

- 8000 proton/iron/nitrogen/helium CORSIKA/CoREAS simulations
- Energy → $10^{18.4}$ to $10^{20.1}$ eV
- Zenith → 65 to 85 deg
- Detector simulation (antenna response unfolded back)

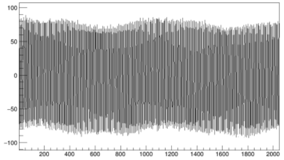


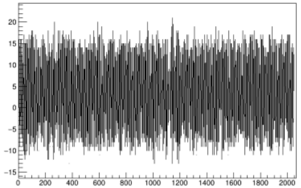
The Radio Detector of Auger is used for practical reasons

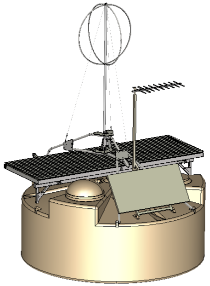
Simulations

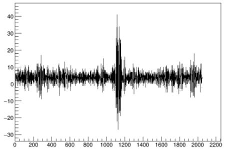
Background traces recorded over one year at the Auger site

Noise Library









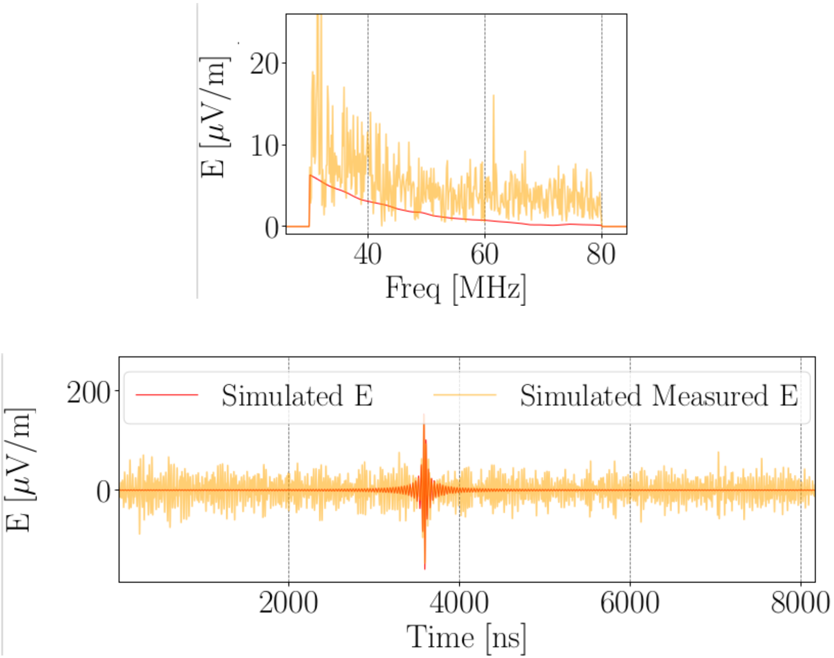
Simulations

Simulated Measurements

Noise Library

RECONSTRUCTION

- Rice method
- Noise subtraction method



Simulated Data Set

Simulations

Simulated Measurements

Rice method

Rice method

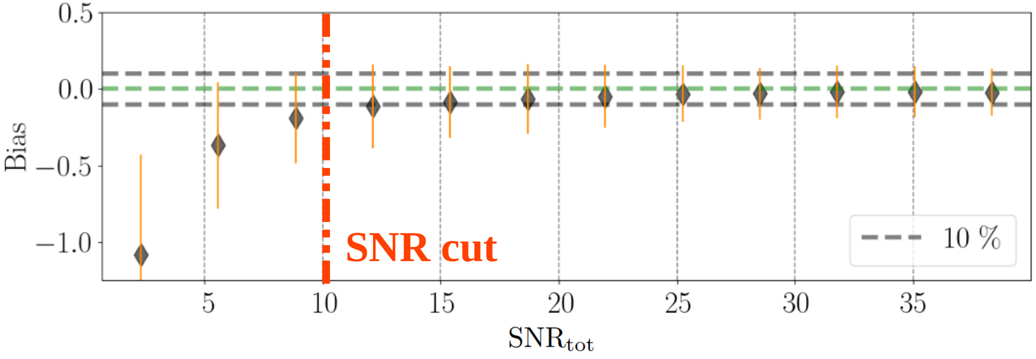
REFERENCE VALUES vs RECONSTRUCTION

Noise subtraction method

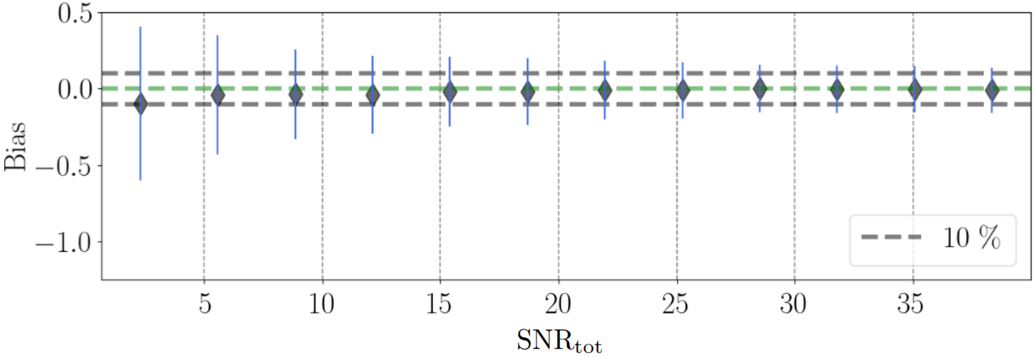
Noise subtraction method

Quality cut: stations affected by thinning artifacts (above 2 Cherenkov radii)

Noise Subtraction Method



Rice-distribution Based Method

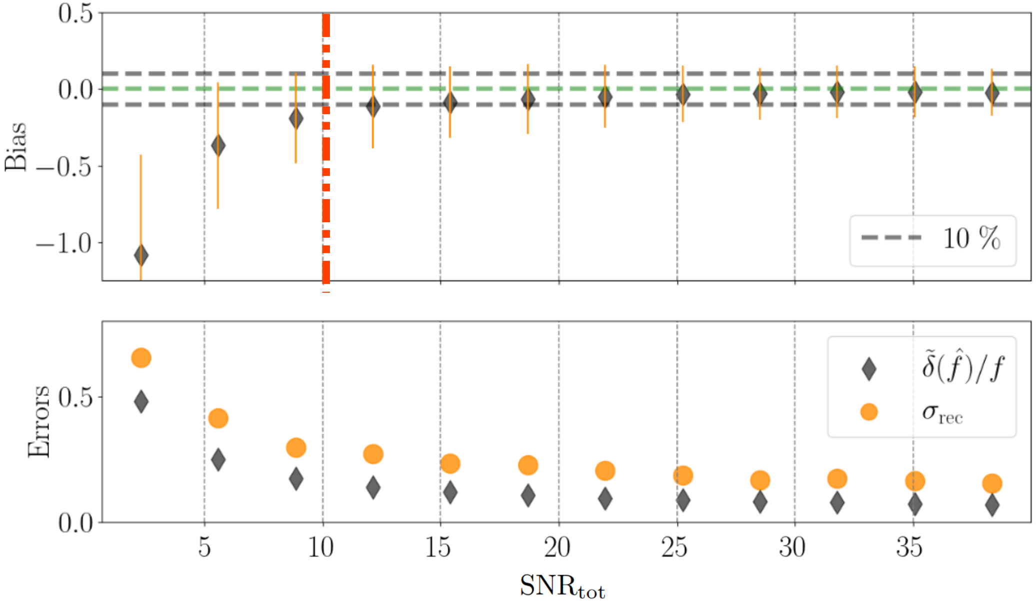


On average the Rice-based method is unbiased even at small SNR

$\psi = \frac{\hat{f}_{\text{pol}}(\vec{r})}{f_{\text{pol}}(\vec{r})}$ $\quad \text{Bias} = \widetilde{\psi} - 1$

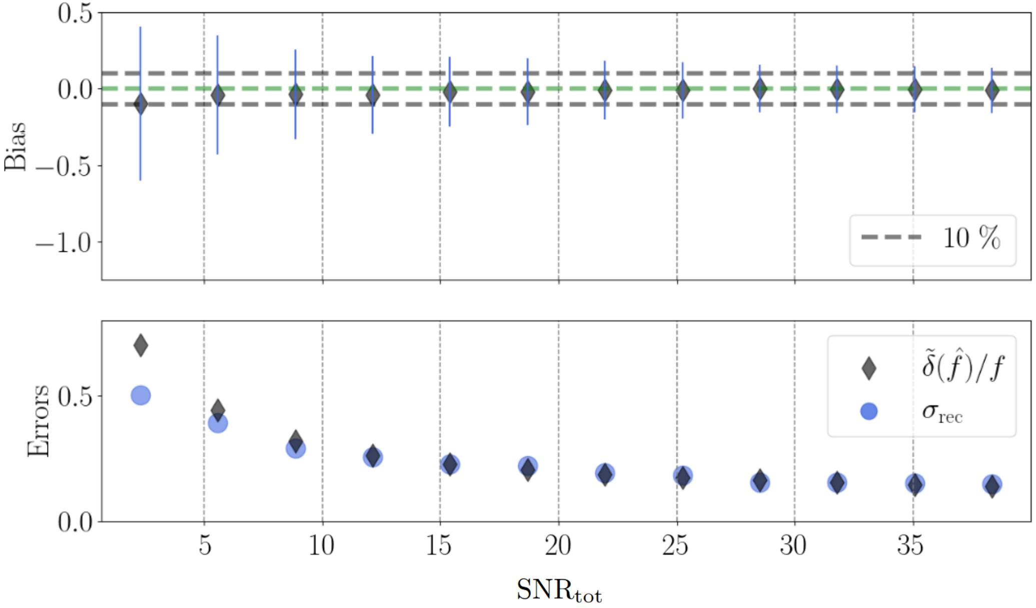
$\text{SNR}_{\text{tot}} = \left(\frac{A_{\text{tot}}^{\text{Hilb}}|_{\max}}{A_{\text{tot}}^{\text{RMS}}} \right)^2$

Noise Subtraction Method



The relative errors are smaller than the reconstruction resolution of the same bin

Rice-distribution Based Method

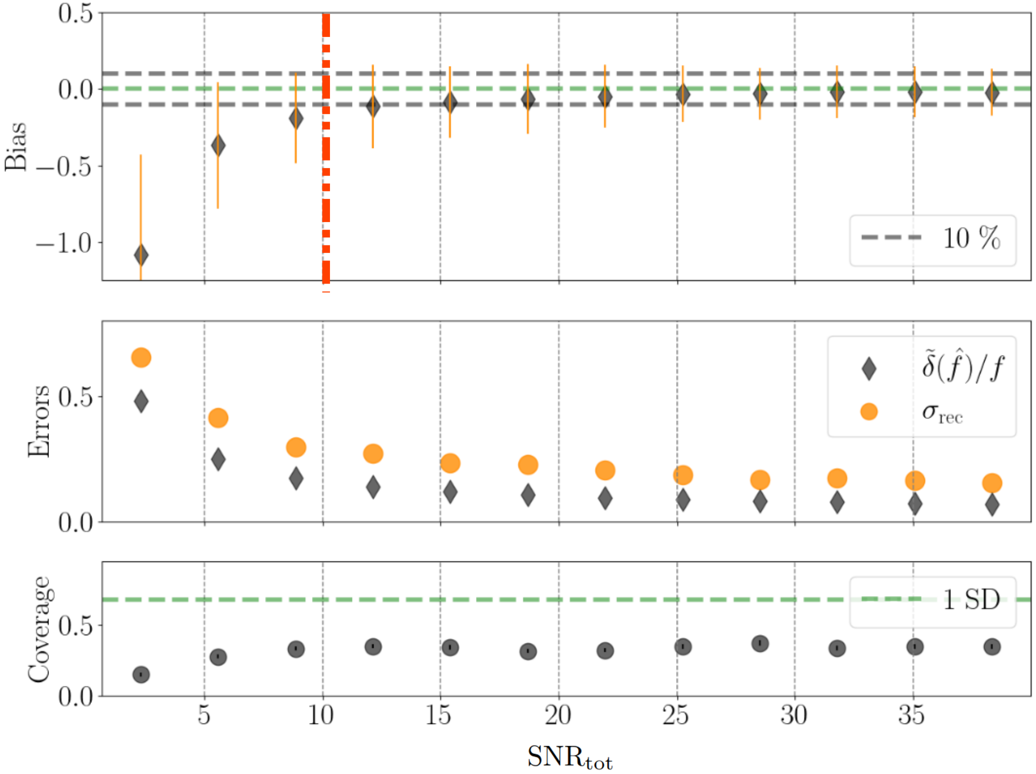


The relative errors of the new method reflect better the reconstruction resolution

$$\sigma_{\text{rec}} \approx \frac{\text{i.q.r.}}{1.35}$$

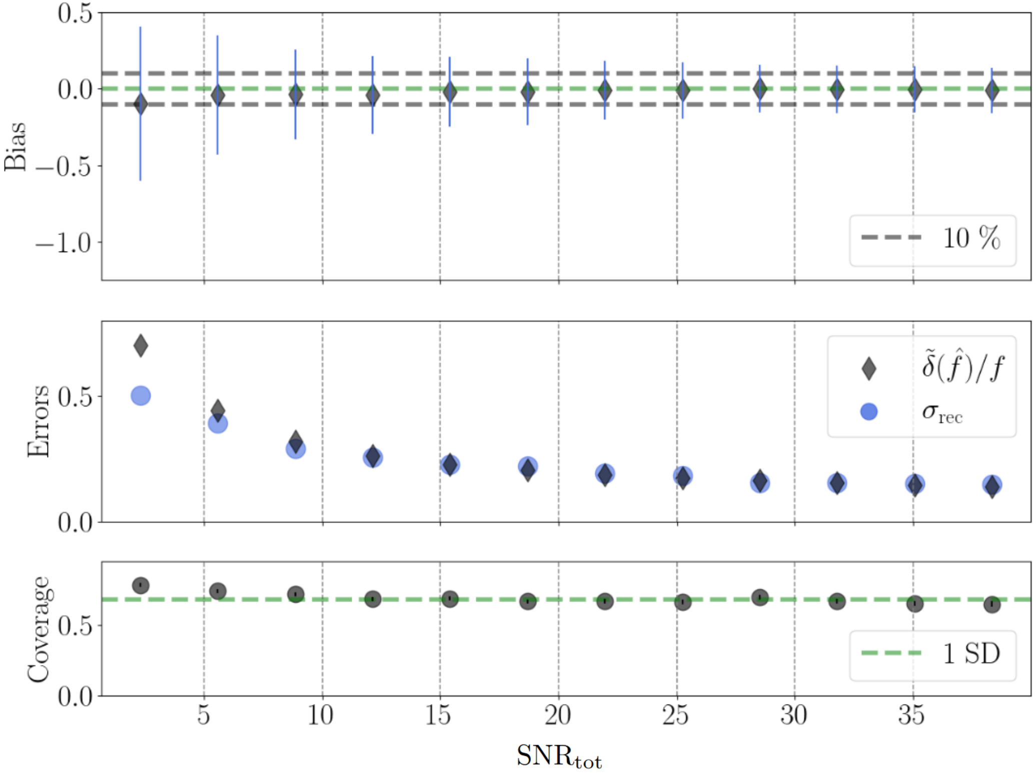
$$\text{SNR}_{\text{tot}} = \left(\frac{|A^{\text{Hilb}}_{\text{tot}}|_{\max}}{A^{\text{RMS}}_{\text{tot}}}\right)^2$$

Noise Subtraction Method



Low coverage

Rice-distribution Based Method



Coverage of the errors fluctuates around 68%

$$\mathrm{SNR}_{\mathrm{tot}} = \left(\frac{A^{\mathrm{Hilb}}_{\mathrm{tot}}|_{\mathrm{max}}}{A^{\mathrm{RMS}}_{\mathrm{tot}}}\right)^2$$

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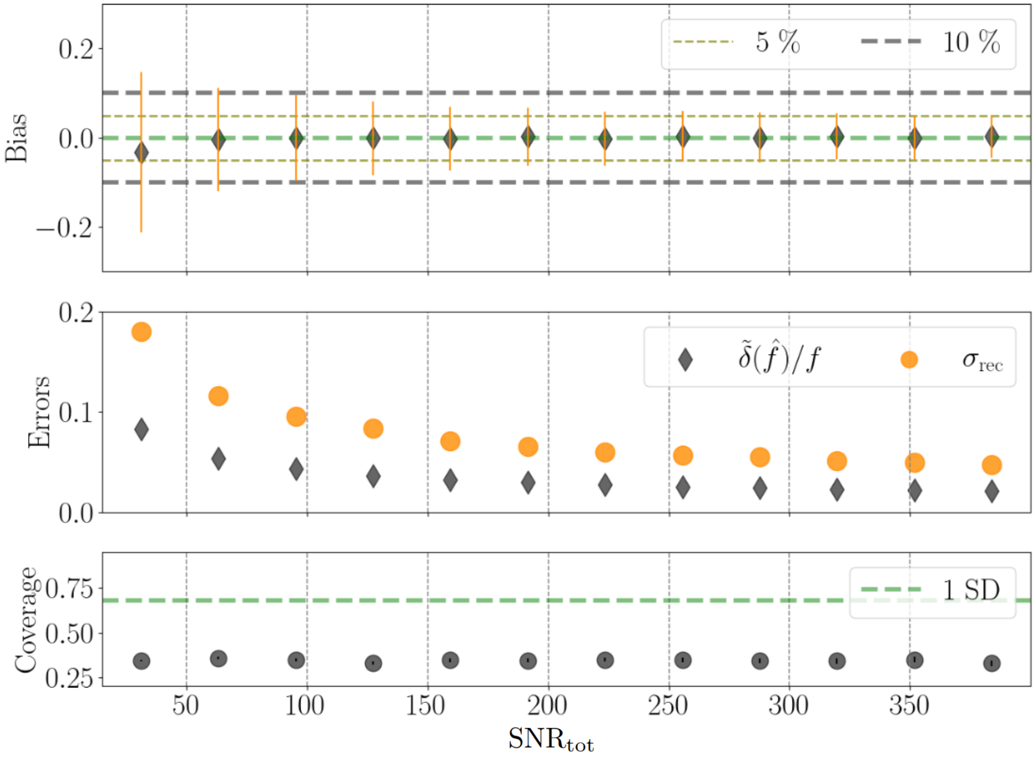
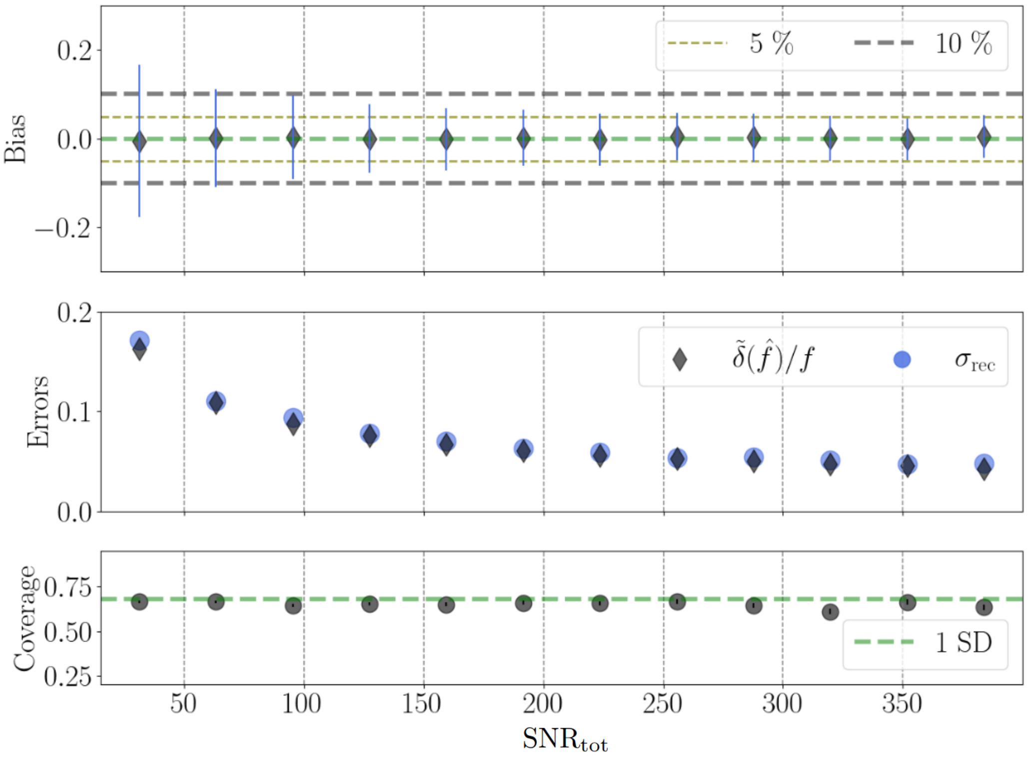
Comparison of the Methods

Noise Subtraction Method



Rice-distribution Based Method



The noise subtraction method underestimates the uncertainties at any SNR.

CAVEAT: SNR<15 excluded in both plots for a fair comparison

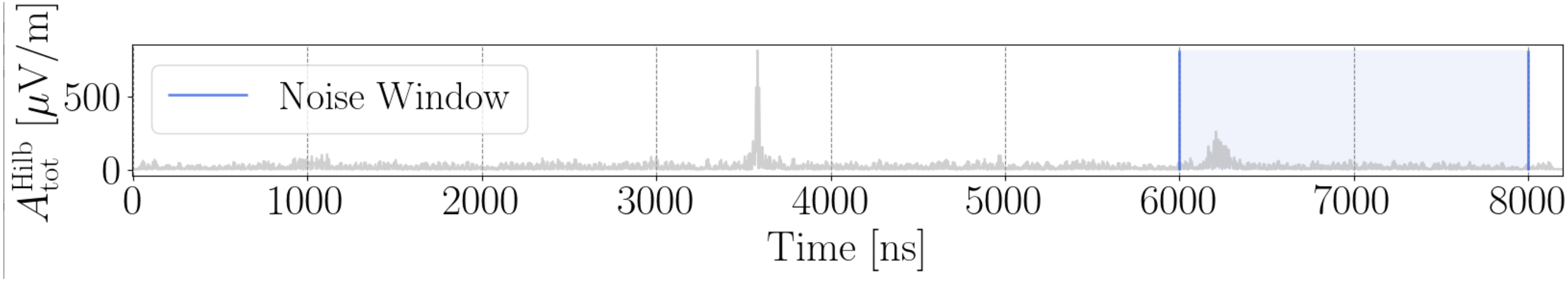
$SNR_{tot} = \left(\frac{A_{tot}^{Hilb}|_{max}}{A_{tot}^{RMS}}\right)^2$

- The fluence estimation based on the Rice distribution shows a smaller bias than the noise subtraction method for small SNR values (on average less than 10%)
- At larger SNR values, the bias of both methods is comparable (on average less than 5%)
- The Rice-distribution method correctly estimate the uncertainties at any SNR (coverage about 68%)
- Paper soon ready for journal submission

Backup

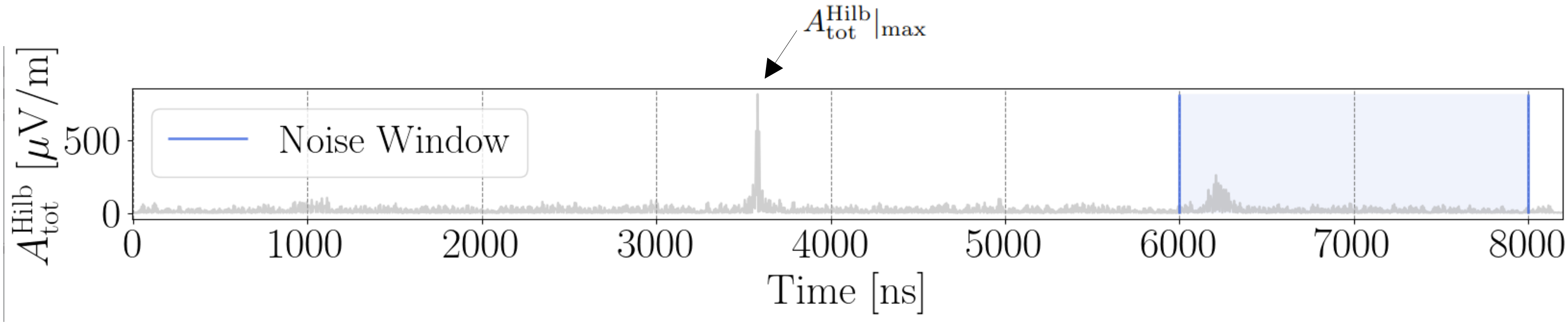
SNR Definition

1. Definition of the trace: $A_{\mathrm{tot}}^{\mathrm{Hilb}}(\vec{r},t) = \sqrt{\sum_{\mathrm{pol}}^{3} \left| E_{\mathrm{pol}}^{\mathrm{Hilb}}(\vec{r},t) \right|^2}$



1. Definition of the trace: $A^{\mathrm{Hilb}}_{\mathrm{tot}}(\vec{r},t) = \sqrt{\sum_{\mathrm{pol}}^{3} \left|E^{\mathrm{Hilb}}_{\mathrm{pol}}(\vec{r},t)\right|^2}$

2. Algorithm to find the maximum of the trace: $A^{\mathrm{Hilb}}_{\mathrm{tot}}|_{\mathrm{max}}$

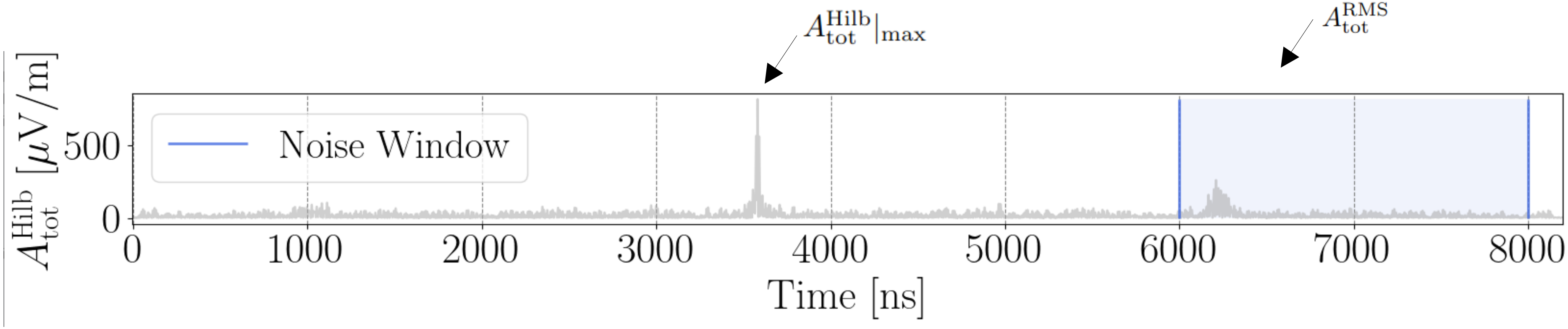


SNR Definition

1. Definition of the trace: $A_{\mathrm{tot}}^{\mathrm{Hilb}}(\vec{r},t) = \sqrt{\sum_{\mathrm{pol}}^{3} \left|E_{\mathrm{pol}}^{\mathrm{Hilb}}(\vec{r},t)\right|^2}$

2. Algorithm to find the maximum of the trace: $A_{\mathrm{tot}}^{\mathrm{Hilb}}|_{\mathrm{max}}$

3. Noise level evaluated in the noise window (RMS): $A_{\mathrm{tot}}^{\mathrm{RMS}} = \sqrt{\sum_{\mathrm{t_j=t_1}}^{t_2} \frac{1}{N}\left(A_{\mathrm{tot}}^{\mathrm{Hilb}}(\vec{r},t_j)\right)^2}$



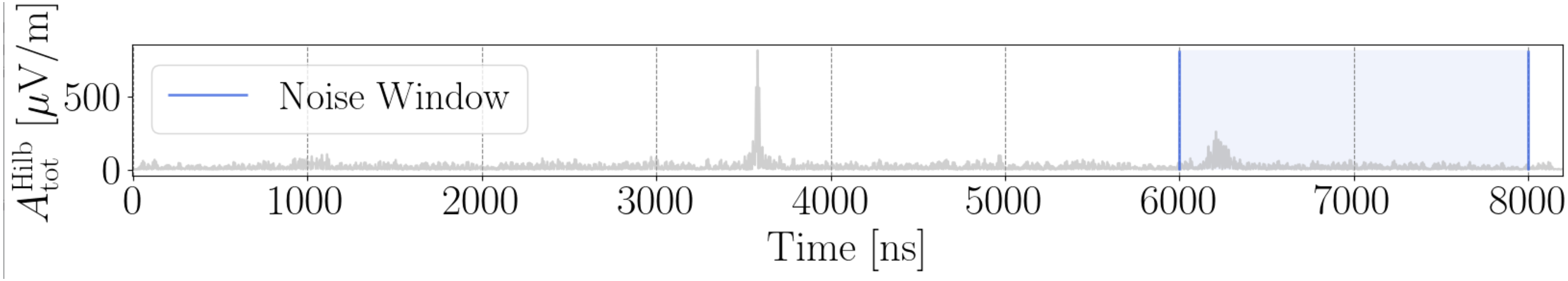
1. Definition of the trace: $A_{\text{tot}}^{\text{Hilb}}(\vec{r},t) = \sqrt{\sum_{\text{pol}}^{3} \left|E_{\text{pol}}^{\text{Hilb}}(\vec{r},t)\right|^2}$

2. Algorithm to find the maximum of the trace: $A_{\text{tot}}^{\text{Hilb}}|_{\text{max}}$

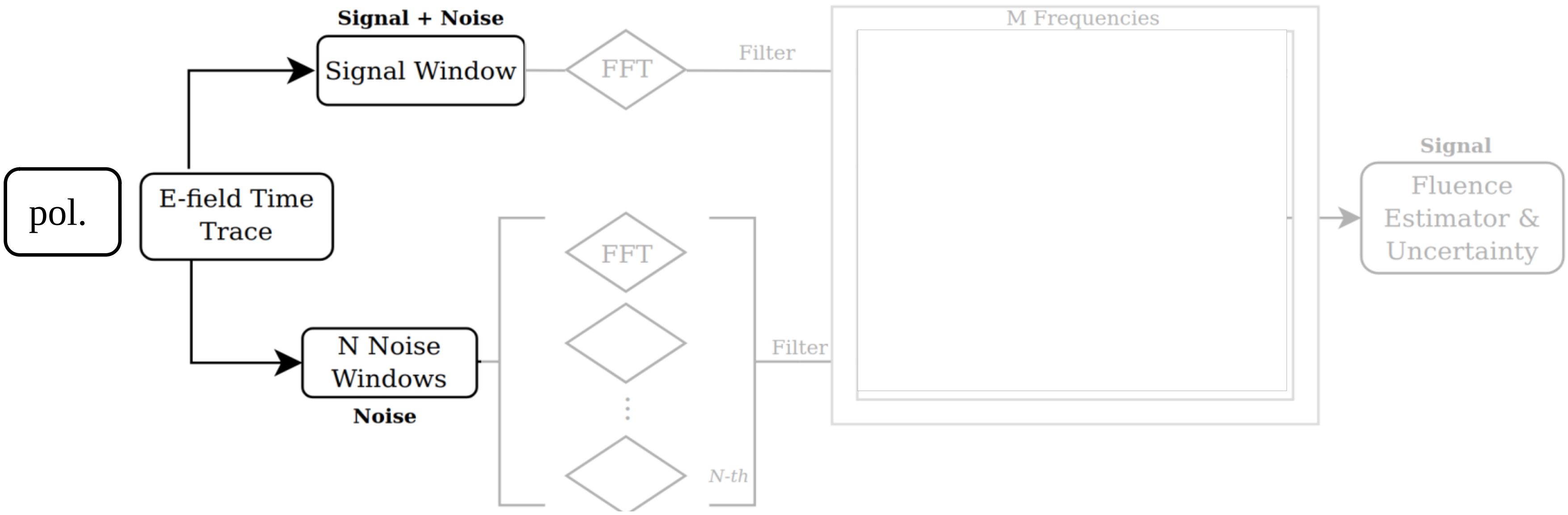
3. Noise level evaluated in the noise window (RMS): $A_{\text{tot}}^{\text{RMS}} = \sqrt{\sum_{\text{t}_\text{j}=\text{t}_1}^{t_2} \frac{1}{N}\left(A_{\text{tot}}^{\text{Hilb}}(\vec{r},t_j)\right)^2}$

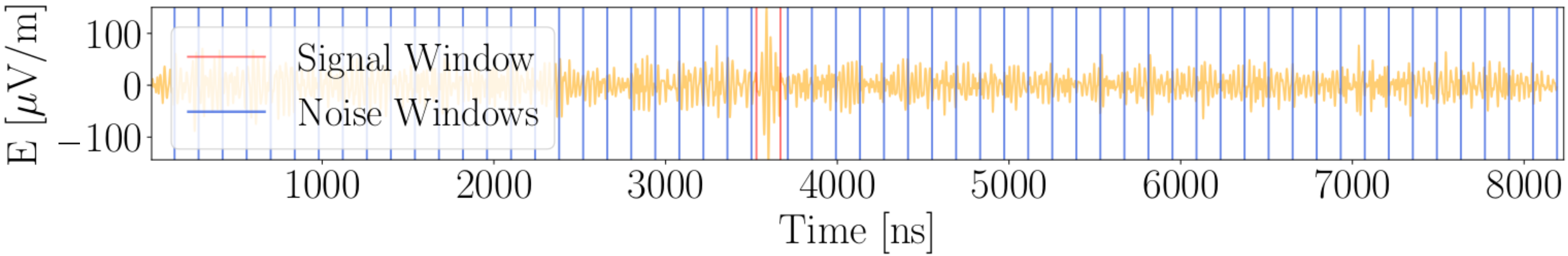
4. Definition of the SNR over all the polarisation: $\text{SNR}_{\text{tot}} = \left(\frac{A_{\text{tot}}^{\text{Hilb}}|_{\text{max}}}{A_{\text{tot}}^{\text{RMS}}}\right)^2$

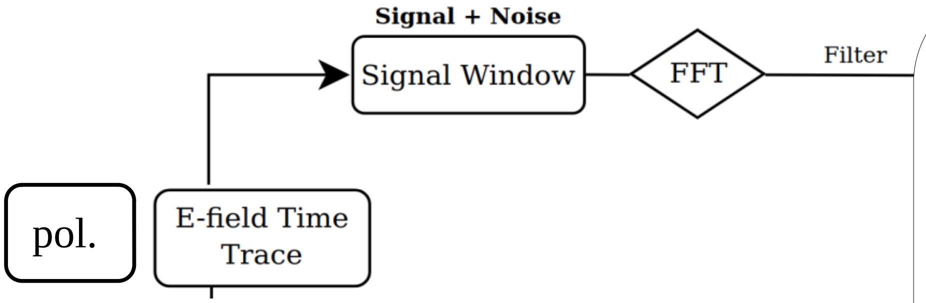
(similar definition at polarisation level)



Rice-distribution Based Method: Derivation



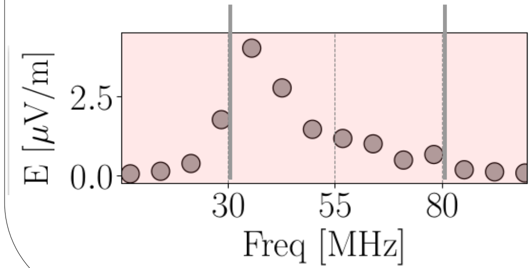


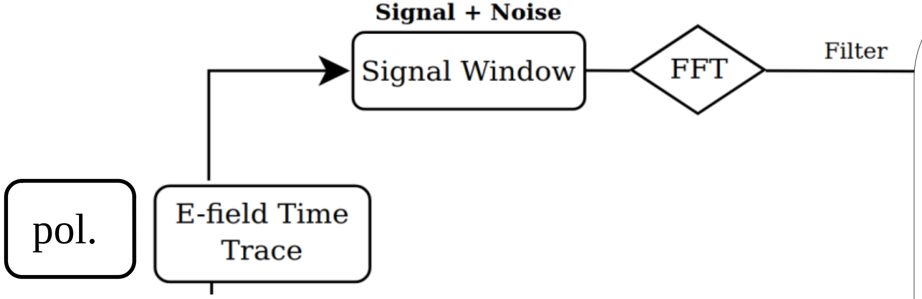


M frequencies

Measured

$$\hat{f}_{\mathrm{a}} = K \sum_{j=0}^{M-1} a^2(v_j) = \sum_{j=0}^{M-1} \hat{f}_{\mathrm{a}}(\nu_j)$$

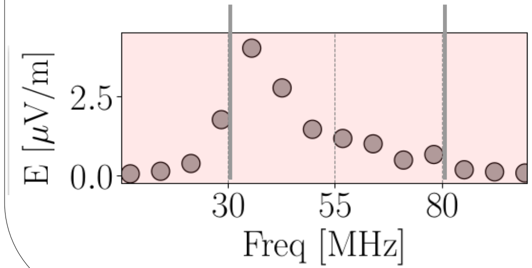




M frequencies

Measured

$$\hat{f}_{\mathrm{a}} = K \sum_{j=0}^{M-1} a^2(v_j) = \sum_{j=0}^{M-1} \hat{f}_{\mathrm{a}}(\nu_j)$$



Unknown

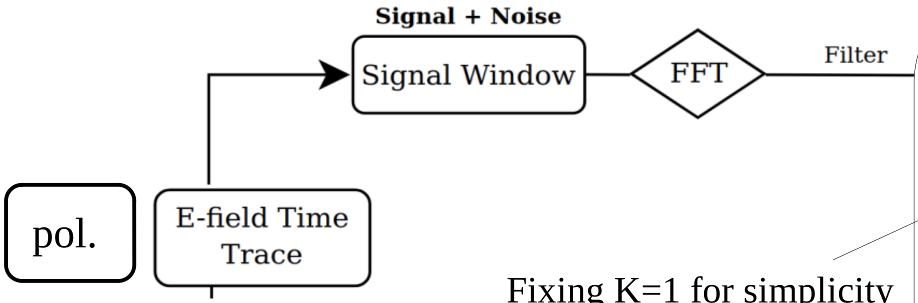
$$f_{\mathrm{s}} = K \sum_{j=0}^{M-1} s^2(v_j) = \sum_{j=0}^{M-1} f_{\mathrm{s}}(\nu_j)$$

Estimator?

$$\hat{f}_{\mathrm{s}} = \sum_{j=0}^{M-1} \hat{f}_{\mathrm{s}}(\nu_j)$$

$$Var(\hat{f}_{\mathrm{s}}) = \sum_{j=0}^{M-1} Var\big(\hat{f}_{\mathrm{s}}(\nu_j)\big)$$

Rice-distribution Based Method: Derivation



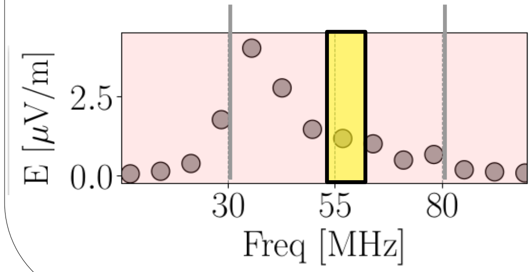
Fixing K=1 for simplicity

j-th frequency

Measured

$$\hat{f}_{\mathrm{a}} = a^2$$

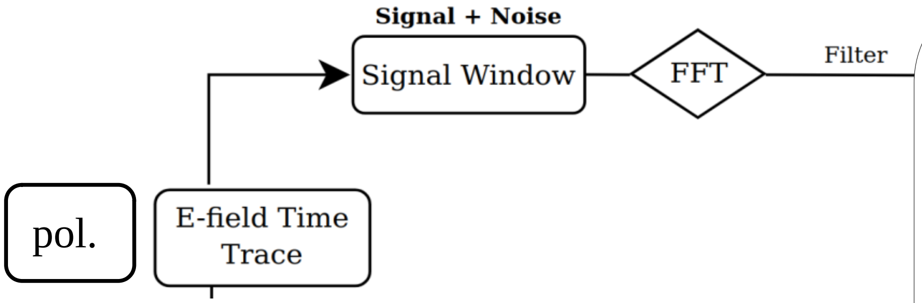
$$a \sim \mathcal{R}ice(s, \sigma)$$



Unknown

$$f_{\mathrm{s}} = s^2$$

Estimator?

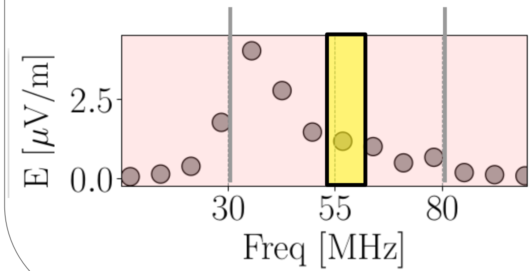


j-th frequency

Measured

$$\hat{f}_{\mathrm{a}} = a^2$$

$$a \sim \mathcal{R}ice(s, \sigma)$$



Unknown

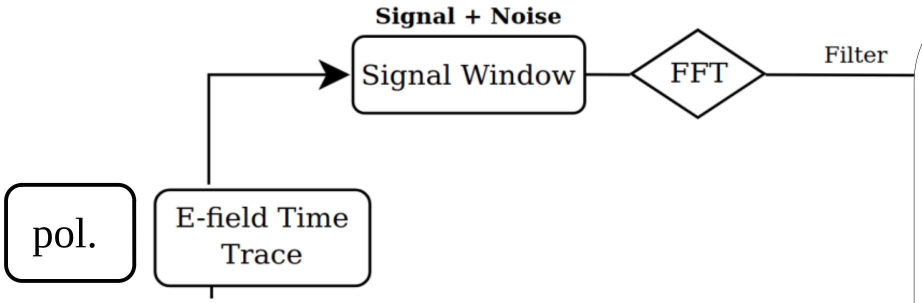
$$f_{\mathrm{s}} = s^2$$

Estimator?

$$\hat{f}_{\mathrm{s}} \propto a^2$$

$$a^2 \sim ?$$

Rice-distribution Based Method: Derivation

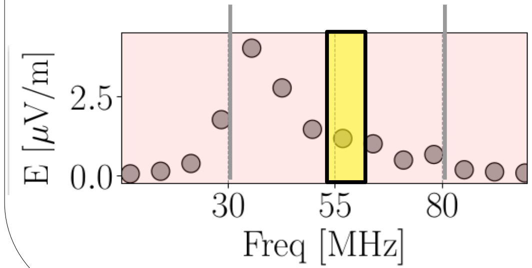


j-th frequency

Measured

$\hat{f}_\mathrm{a} = a^2$

$a \sim \mathcal{R}ice(s, \sigma)$



Unknown

$f_\mathrm{s} = s^2$

Estimator?

$\hat{f}_\mathrm{s} \propto a^2$

$a^2 \sim ?$

Derivation:

$b = a/\sigma$	Change of variable
σ	Noise at bin level

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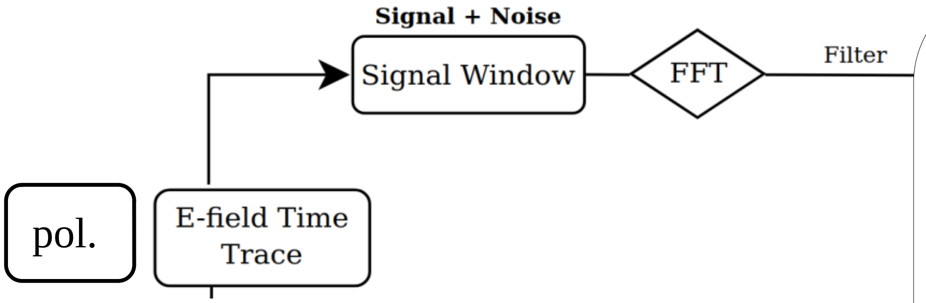
Rice-distribution Based Method: Derivation

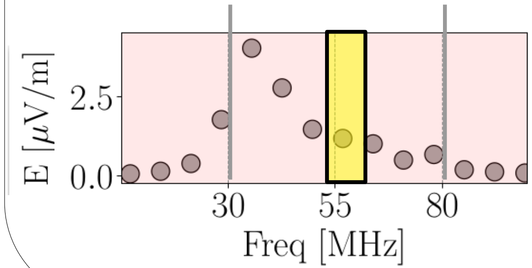


j-th frequency

Measured

$\hat{f}_{\mathrm{a}} = a^2$

$a \sim \mathcal{R}ice(s, \sigma)$



Unknown

$f_{\mathrm{s}} = s^2$

Estimator?

$\hat{f}_{\mathrm{s}} \propto a^2$

$a^2 \sim ?$

Derivation:

$b = a/\sigma$ — Change of variable → $b \sim \mathcal{R}ice(s/\sigma, 1) = \chi_{\mathrm{nc}}(DF = 2, \lambda = s/\sigma)$

σ — Noise at bin level

Degrees of Freedom (DF)

Non-centrality parameter (λ)

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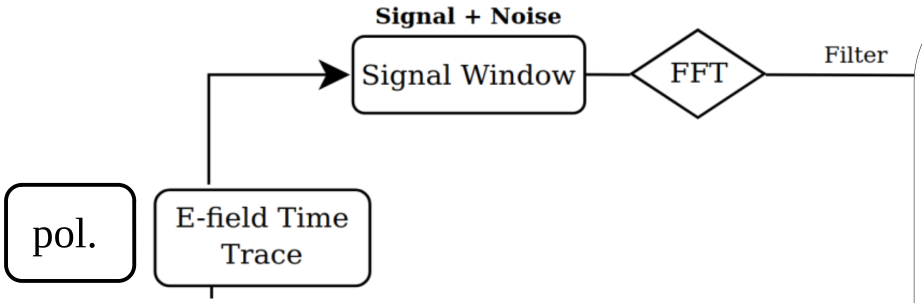
Rice-distribution Based Method: Derivation

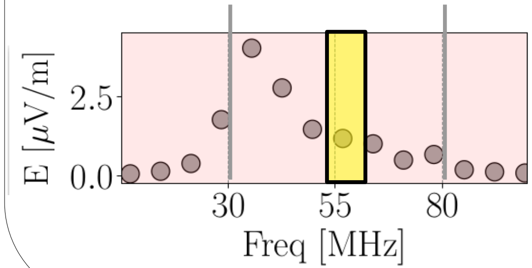


j-th frequency

Measured

$\hat{f}_{\mathrm{a}} = a^2$

$a \sim \mathcal{R}ice(s, \sigma)$



Unknown

$f_{\mathrm{s}} = s^2$

Estimator?

$\hat{f}_{\mathrm{s}} \propto a^2$

$a^2 \sim ?$

Derivation:

$b = a/\sigma$ — Change of variable → $b^2 \sim \chi_{\mathrm{nc}}(DF = 2, \lambda = s^2/\sigma^2)$

σ — Noise at bin level

$\mathrm{E}(b^2) = 2 + (s/\sigma)^2$ — Expected value

$\mathrm{Var}(b^2) = 2\ \left(2 + 2\,(s/\sigma)^2\right)$ — Variance

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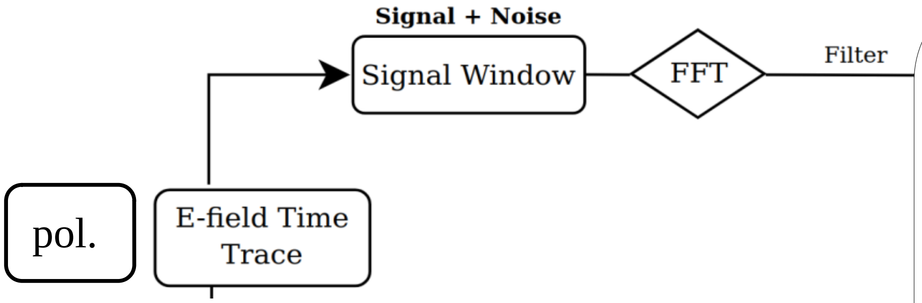


j-th frequency

Measured

$\hat{f}_a = a^2$

$a \sim \mathcal{R}ice(s, \sigma)$

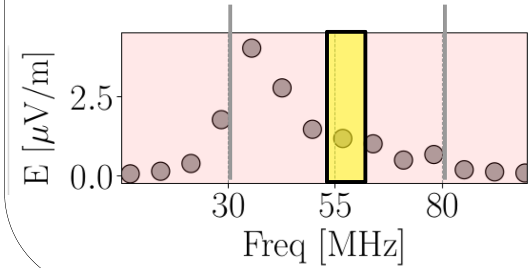
Unknown

$f_s = s^2$

Estimator?

$\hat{f}_s \propto a^2$

$a^2 \sim ?$



Derivation:

$$\mathrm{E}(b^2) = 2 + (s/\sigma)^2$$

$$\mathrm{Var}(b^2) = 2\ \left(2 + 2\,(s/\sigma)^2\right)$$

$$a^2 = \sigma^2\, b^2 \longrightarrow$$

$$\mathrm{E}\left(\hat{f}_a\right) = \mathrm{E}(a^2) = \sigma^2\,\mathrm{E}(b^2)$$ Expected value

$$\mathrm{Var}\left(\hat{f}_a\right) = \mathrm{Var}(a^2) = \sigma^4\,\mathrm{Var}(b^2)$$ Variance

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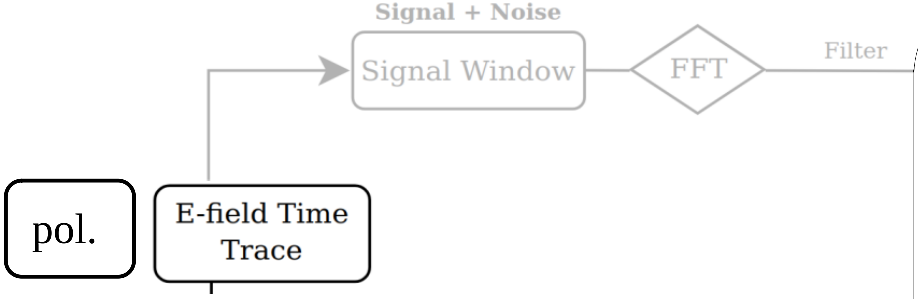
Rice-distribution Based Method: Derivation

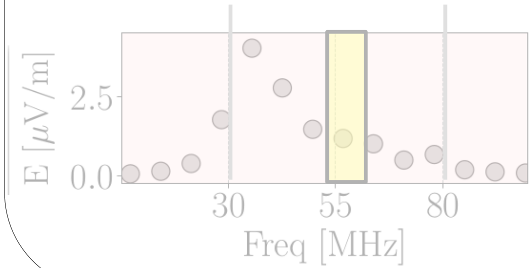


j-th frequency

Measured

$$\hat{f}_\mathrm{a} = a^2$$

$$a \sim \mathcal{R}ice(s, \sigma)$$



Unknown

$$f_\mathrm{s} = s^2$$

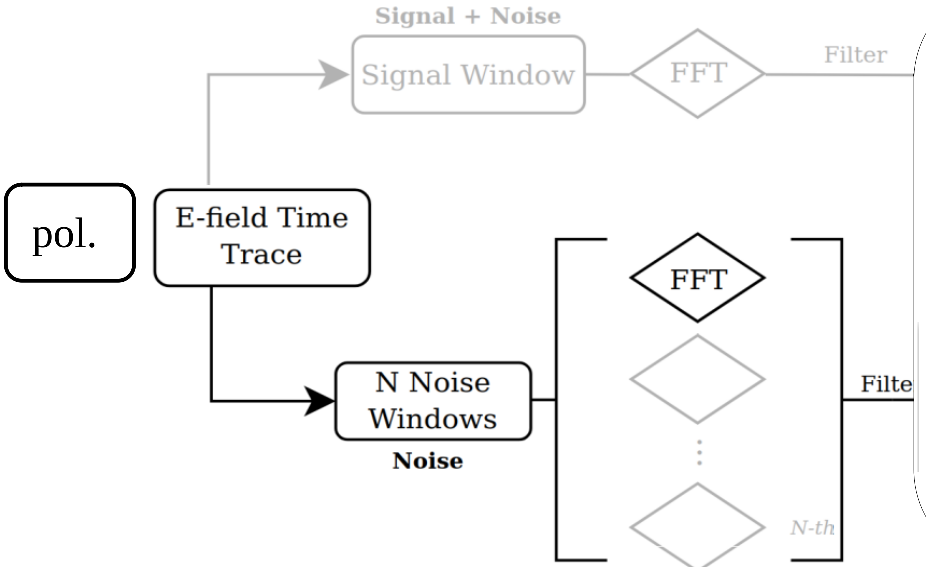
Estimator?

$$\mathrm{E}\left(\hat{f}_a\right) = 2\,\sigma^2 + s^2$$

$$\mathrm{Var}\left(\hat{f}_a\right) = 4\,\sigma^2\,(\sigma^2 + 2\,s^2)$$

$2\,\sigma^2$?

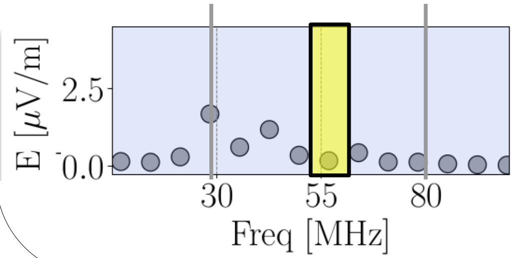
Rice-distribution Based Method: Derivation



j-th frequency

Measured

$n_i \sim \; Rayleigh(\sigma)$



Unknown

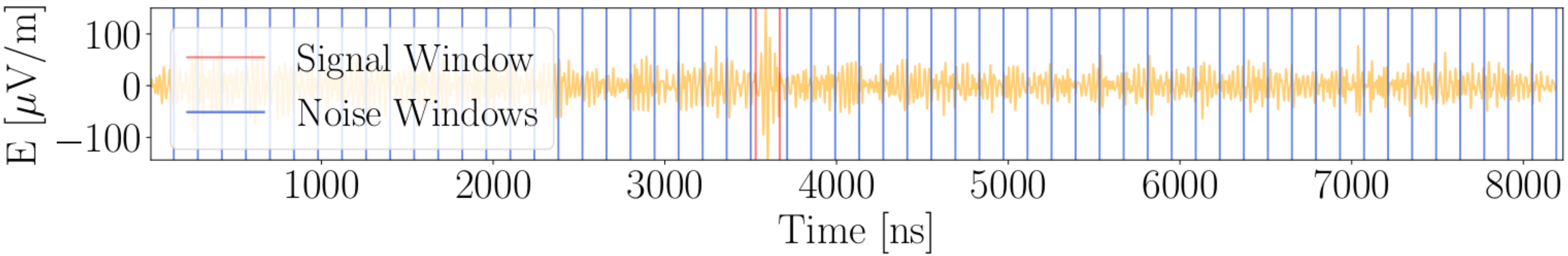
$f_s = s^2$

Estimator?

$$\mathrm{E}\left(\hat{f}_a\right) = 2\,\sigma^2 + s^2$$

$$\mathrm{Var}\left(\hat{f}_a\right) = 4\,\sigma^2\left(\sigma^2 + 2\,s^2\right)$$

$2\,\sigma^2$?



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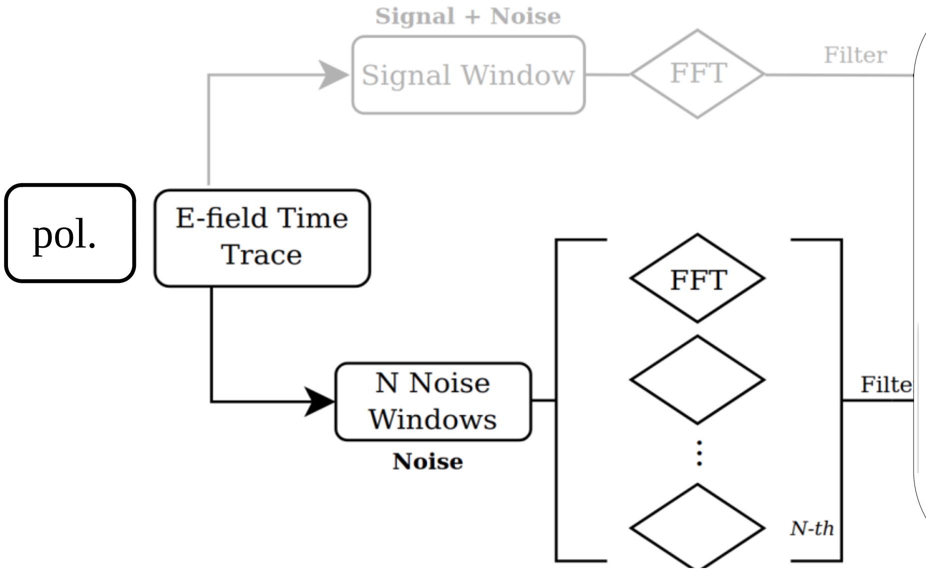
Rice-distribution Based Method: Derivation

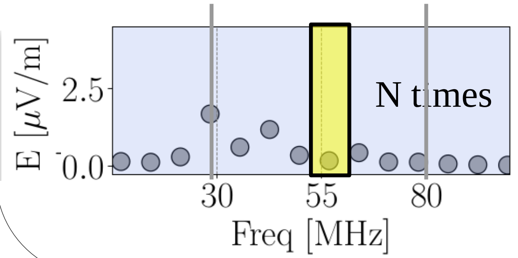


j-th frequency

Measured

$n_i \sim \; Rayleigh(\sigma)$

$\hat{f}_\mathrm{n} = \frac{1}{N} \sum_{i=0}^{N-1} n_i^2$ sample mean



Unknown

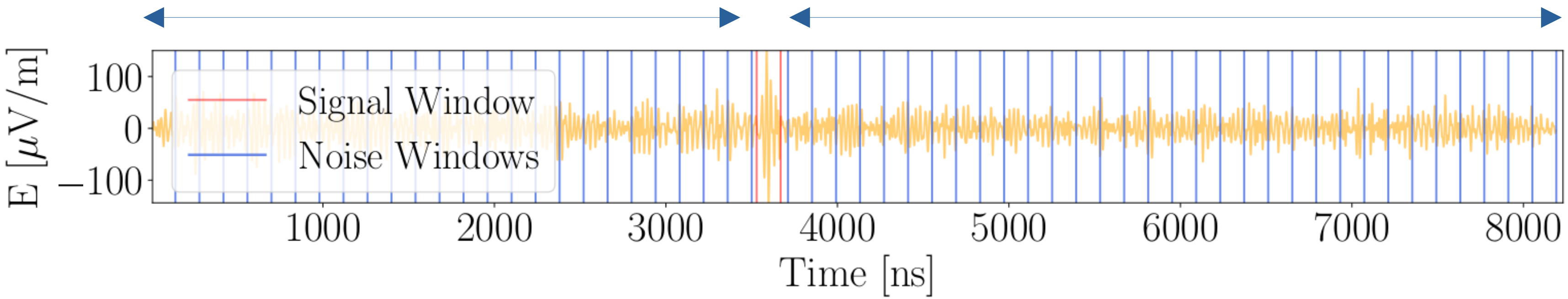
$f_\mathrm{s} = s^2$

Estimator?

$\mathrm{E}\left(\hat{f}_a\right) = 2\,\sigma^2 + s^2$

$\mathrm{Var}\left(\hat{f}_a\right) = 4\,\sigma^2\,(\sigma^2 + 2\,s^2)$

$2\,\sigma^2$?



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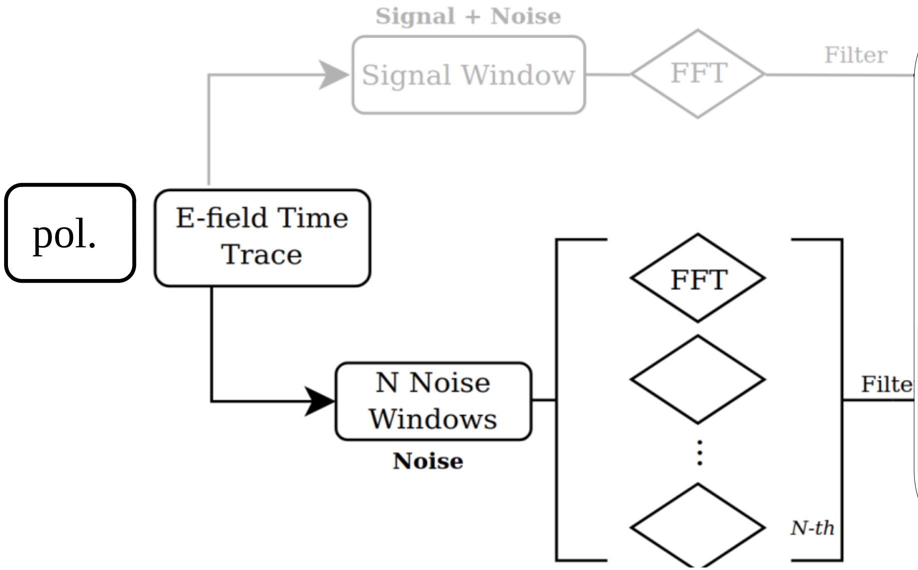
Rice-distribution Based Method: Derivation

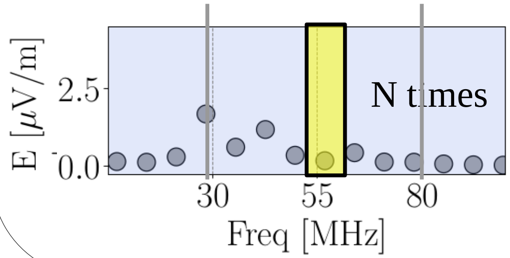


j-th frequency

Measured

$$n_i \sim \ Rayleigh(\sigma)$$

$$\hat{f}_{\rm n} = \frac{1}{N}\sum_{i=0}^{N-1} n_i^2$$



Unknown

$$f_{\rm s} = s^2$$

Estimator?

$$\mathrm{E}\left(\hat{f}_a\right) = 2\,\sigma^2 + s^2$$

$$\mathrm{Var}\left(\hat{f}_a\right) = 4\,\sigma^2\,(\sigma^2 + 2\,s^2)$$

$2\,\sigma^2$?

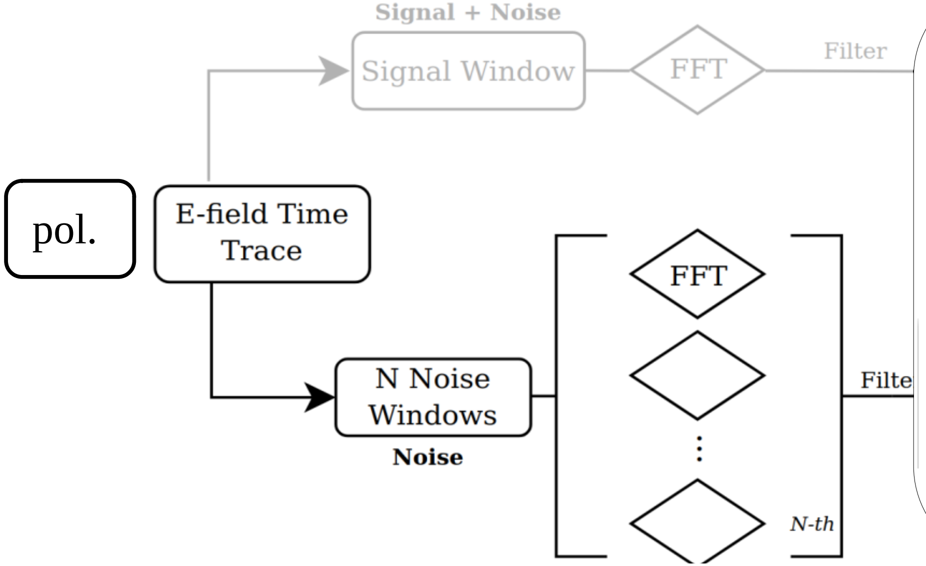
Derivation:

i-th window:

$$n_i \sim \ Rayleigh(\sigma)$$

Change of variable

$$(n_i/\sigma)^2 \sim \ \chi^2(DF = 2)$$

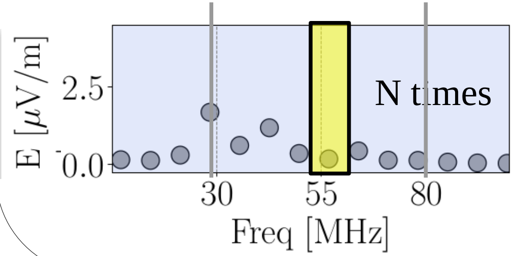


j-th frequency

Measured

$$n_i \sim \ Rayleigh(\sigma)$$

$$\hat{f}_\mathrm{n} = \frac{1}{N}\sum_{i=0}^{N-1} n_i^2$$



Unknown

$$f_\mathrm{s} = s^2$$

Estimator?

$$\mathrm{E}\left(\hat{f}_a\right) = 2\,\sigma^2 + s^2$$

$$\mathrm{Var}\left(\hat{f}_a\right) = 4\,\sigma^2\,(\sigma^2 + 2\,s^2)$$

$2\,\sigma^2$?

Derivation:

i-th window:

$$n_i \sim \ Rayleigh(\sigma)$$

Change of variable

$$(n_i/\sigma)^2 \sim \ \chi^2(DF = 2)$$

→

Definition of a new variable over the windows:

$$T = \sum_{i=0}^{N-1}(n_i/\sigma)^2 \sim \ \chi^2(DF = 2\cdot N) \approx \mathcal{N}(\mu = 2\,N, SD = 2\sqrt{N})$$

normally distributed (N large)

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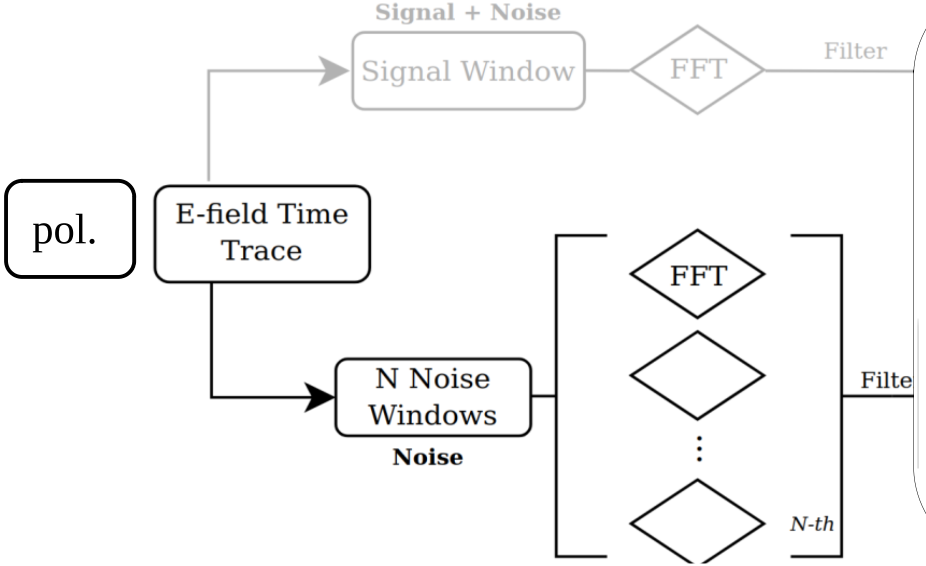
Rice-distribution Based Method: Derivation

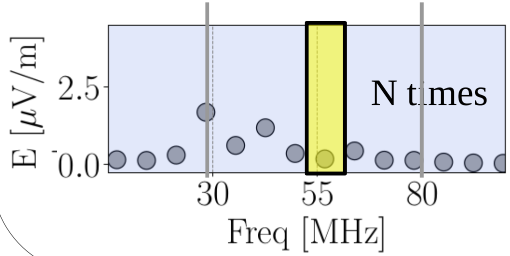


j-th frequency

Measured

$$n_i \sim \; Rayleigh(\sigma)$$

$$\hat{f}_{\mathrm{n}} = \frac{1}{N}\sum_{i=0}^{N-1} n_i^2$$



Unknown

$$f_{\mathrm{s}} = s^2$$

Estimator?

$$\mathrm{E}\left(\hat{f}_a\right) = 2\,\sigma^2 + s^2$$

$$\mathrm{Var}\left(\hat{f}_a\right) = 4\,\sigma^2\left(\sigma^2 + 2\,s^2\right)$$

$2\,\sigma^2$?

Derivation:

$$\hat{f}_{\mathrm{n}} = \frac{1}{N}\sum_{i=0}^{N-1} n_i^2$$

$$T = \sum_{i=0}^{N-1}(n_i/\sigma)^2 \sim \mathcal{N}(\mu = 2\,N, SD = 2\sqrt{N})$$

→

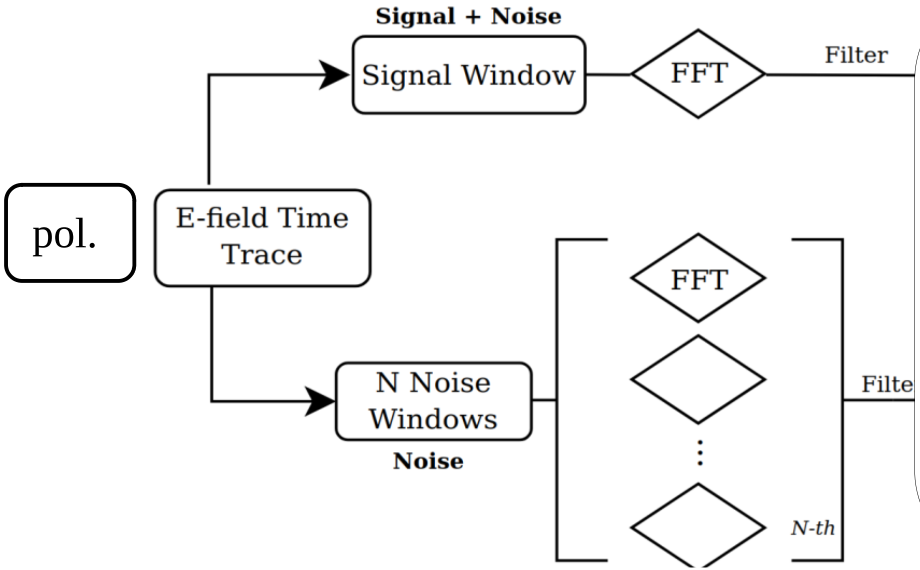
normally distributed

$$\hat{f}_{\mathrm{n}} = \frac{\sigma^2}{N}T \sim \mathcal{N}(\mu = 2\sigma^2, SD = 2\sigma^2/\sqrt{N})$$

Unbiased estimator of $2\,\sigma^2 := f_n$

Sample standard deviation can be neglected

Rice-distribution Based Method: Derivation



j-th frequency

Measured

$$a \sim \mathcal{Rice}(s, \sigma)$$

$$\hat{f}_{\mathrm{a}} = a^2$$

$$n_i \sim \ Rayleigh(\sigma)$$

$$\hat{f}_{\mathrm{n}} = \frac{1}{N}\sum_{i=0}^{N-1} n_i^2$$

Unknown

$$f_{\mathrm{s}} = s^2$$

Estimator?

$$\mathrm{E}\left(\hat{f}_a\right) = 2\,\sigma^2 + s^2$$

$$\mathrm{Var}\left(\hat{f}_a\right) = 4\,\sigma^2\left(\sigma^2 + 2\,s^2\right)$$

$$\mathrm{E}\left(\hat{f}_n\right) = 2\,\sigma^2 := f_n$$

Derivation:

$$\hat{f}_s = \hat{f}_a - \hat{f}_n$$

$$\mathrm{E}\left(\hat{f}_s\right) = \mathrm{E}\left(\hat{f}_a\right) - \mathrm{E}\left(\hat{f}_n\right) = s^2 = f_s$$

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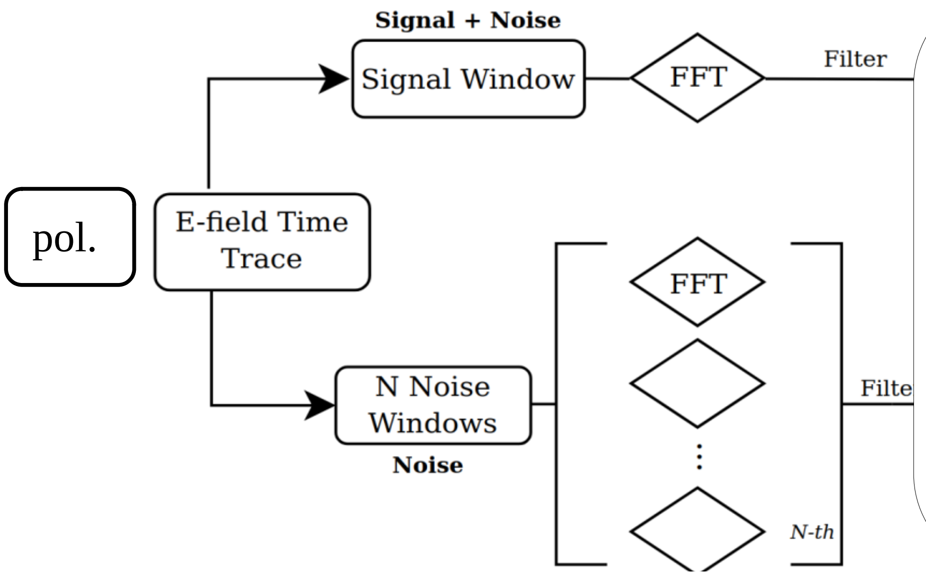
Rice-distribution Based Method: Derivation



j-th frequency

Measured

$$a \sim \mathcal{R}ice(s, \sigma)$$

$$\hat{f}_{\mathrm{a}} = a^2$$

$$n_i \sim \; Rayleigh(\sigma)$$

$$\hat{f}_{\mathrm{n}} = \frac{1}{N} \sum_{i=0}^{N-1} n_i^2$$

Unknown

$$f_{\mathrm{s}} = s^2$$

Estimator?

$$\mathrm{E}\left(\hat{f}_a\right) = 2\,\sigma^2 + s^2$$

$$\mathrm{Var}\left(\hat{f}_a\right) = 4\,\sigma^2\,(\sigma^2 + 2\,s^2)$$

$$\mathrm{E}\left(\hat{f}_n\right) = 2\,\sigma^2 := f_n$$

Derivation:

$$\hat{f}_s = \hat{f}_a - \hat{f}_n$$

$$\mathrm{E}\left(\hat{f}_s\right) = \mathrm{E}\left(\hat{f}_a\right) - \mathrm{E}\left(\hat{f}_n\right) = s^2 = f_s$$

Physical boundary →

$$\hat{f}_s = \begin{cases} \hat{f}_a - \hat{f}_n \text{ if } \hat{f}_a \geq \hat{f}_n \\ 0 \text{ if } \hat{f}_a < \hat{f}_n \end{cases}$$

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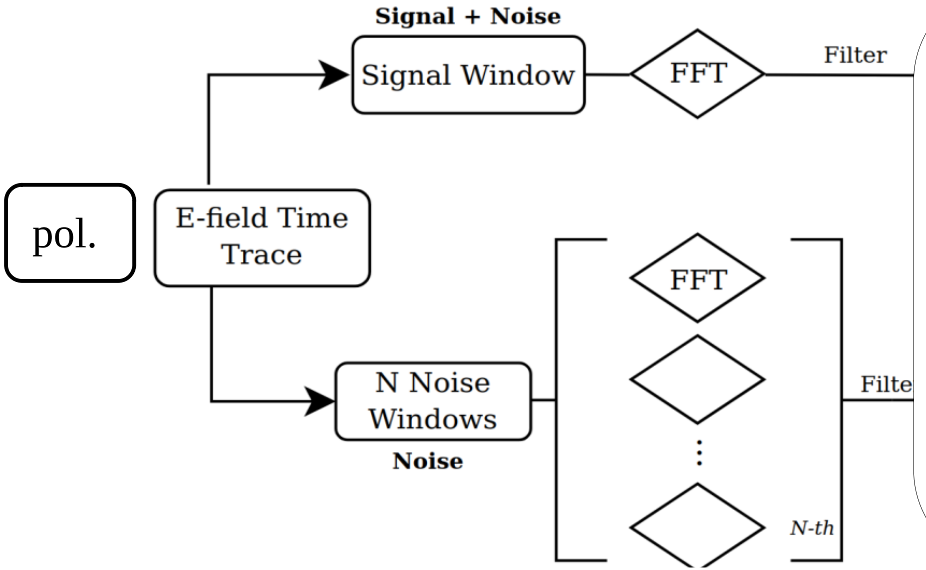
Rice-distribution Based Method: Derivation



j-th frequency

Measured

$$a \sim \mathcal{Rice}(s, \sigma)$$

$$\hat{f}_{\mathrm{a}} = a^2$$

$$n_i \sim \ \mathcal{Rayleigh}(\sigma)$$

$$\hat{f}_{\mathrm{n}} = \frac{1}{N} \sum_{i=0}^{N-1} n_i^2$$

Unknown

$$f_{\mathrm{s}} = s^2$$

Estimator?

$$\mathrm{E}\left(\hat{f}_a\right) = 2\,\sigma^2 + s^2$$

$$\mathrm{Var}\left(\hat{f}_a\right) = 4\,\sigma^2\left(\sigma^2 + 2\,s^2\right)$$

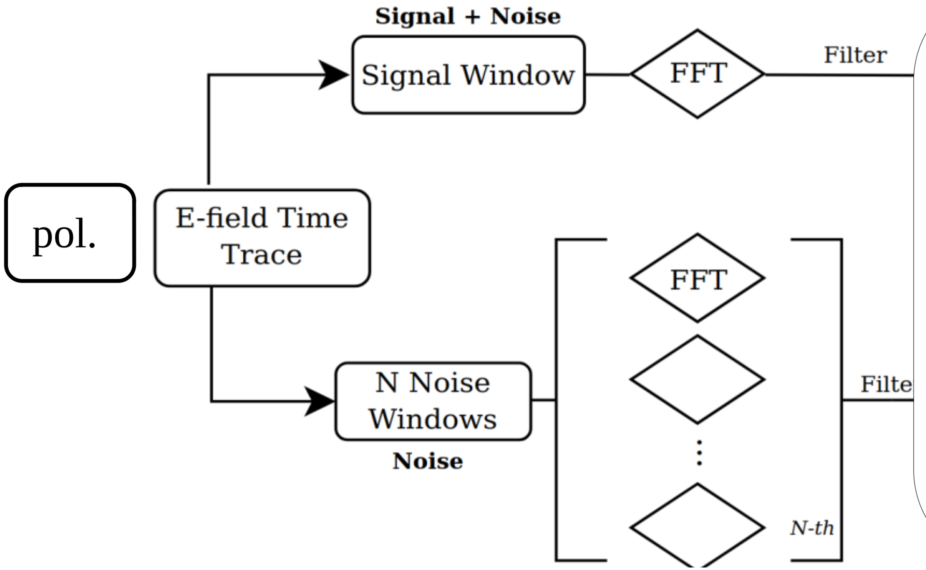
$$\mathrm{E}\left(\hat{f}_n\right) = 2\,\sigma^2 := f_n$$

Derivation:

$$\hat{f}_s = \begin{cases} \hat{f}_a - \hat{f}_n \text{ if } \hat{f}_a \geq \hat{f}_n \\ 0 \text{ if } \hat{f}_a < \hat{f}_n \end{cases}$$

$$\mathrm{Var}(\hat{f}_s) = \ \mathrm{Var}\left(\hat{f}_a\right) = 4\,\sigma^2\left(\sigma^2 + 2\,s^2\right)$$

Rice-distribution Based Method: Derivation



j-th frequency

Measured

$$a \sim \mathcal{R}ice(s, \sigma)$$

$$\hat{f}_{\text{a}} = a^2$$

$$n_i \sim \ Rayleigh(\sigma)$$

$$\hat{f}_{\text{n}} = \frac{1}{N} \sum_{i=0}^{N-1} n_i^2$$

Unknown

$$f_{\text{s}} = s^2$$

Estimator?

$$\mathrm{E}\left(\hat{f}_a\right) = 2\,\sigma^2 + s^2$$

$$\mathrm{Var}\left(\hat{f}_a\right) = 4\,\sigma^2\,(\sigma^2 + 2\,s^2)$$

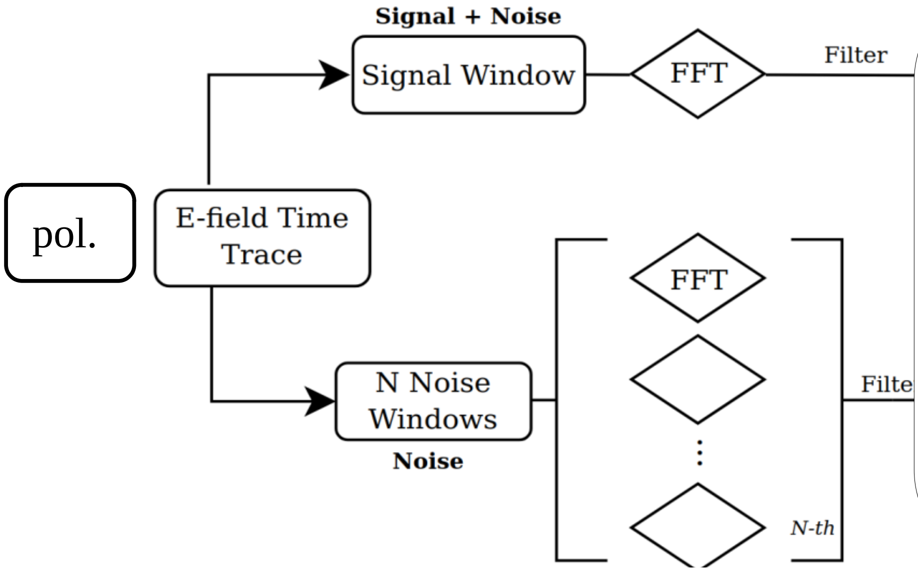
$$\mathrm{E}\left(\hat{f}_n\right) = 2\,\sigma^2 := f_n$$

Derivation:

$$\hat{f}_s = \begin{cases} \hat{f}_a - \hat{f}_n \text{ if } \hat{f}_a \geq \hat{f}_n \\ 0 \text{ if } \hat{f}_a < \hat{f}_n \end{cases}$$

$$\mathrm{Var}(\hat{f}_s) = 4\,\sigma^2\,(\sigma^2 + 2\,s^2) \xrightarrow[2\,\sigma^2 \approx \hat{f}_n]{s^2 \approx \hat{f}_s} \mathrm{Var}(\hat{f}_s) \approx \hat{f}_n \left(\hat{f}_n + 4\,\hat{f}_s\right)$$

Rice-distribution Based Method: Derivation



j-th frequency

Measured

$$a \sim \mathcal{R}ice(s, \sigma)$$

$$\hat{f}_{\mathrm{a}} = a^2$$

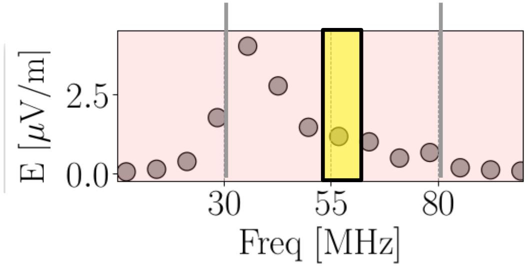
$$n_i \sim \; Rayleigh(\sigma)$$

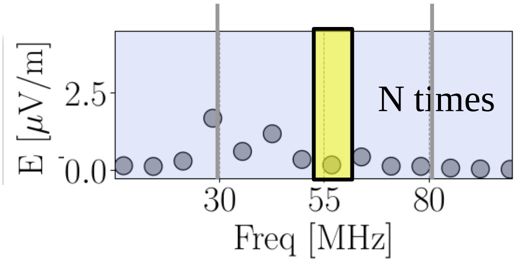
$$\hat{f}_{\mathrm{n}} = \frac{1}{N} \sum_{i=0}^{N-1} n_i^2$$

Estimator & Uncertainty

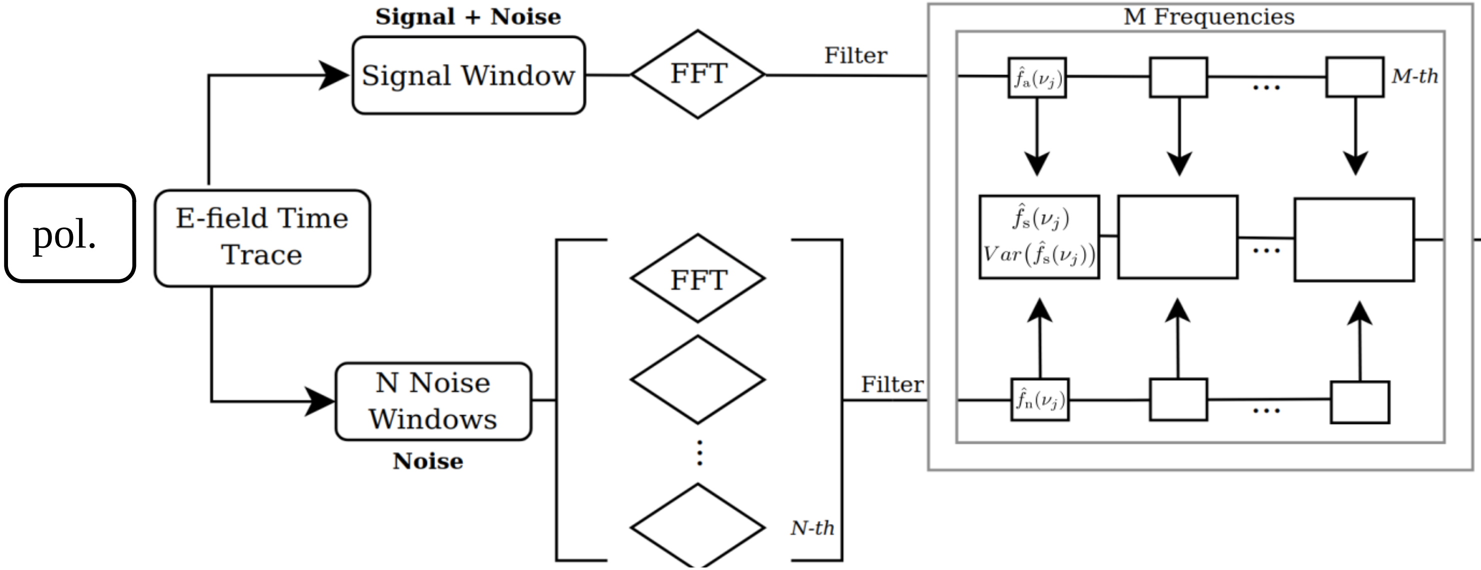
$$\hat{f}_s = \begin{cases} \hat{f}_a - \hat{f}_n \text{ if } \hat{f}_a \geq \hat{f}_n \\ 0 \text{ if } \hat{f}_a < \hat{f}_n \end{cases}$$

$$\delta f_s = \sqrt{\hat{f}_n \left(\hat{f}_n + 4\, \hat{f}_s\right)}$$



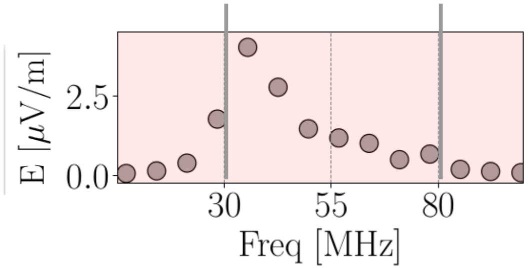


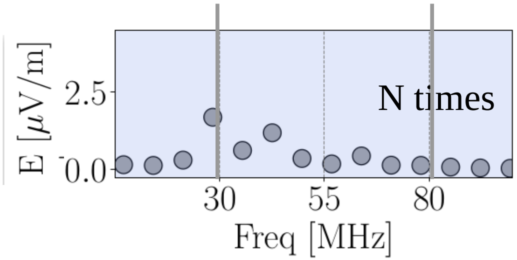
Rice-distribution Based Method: Derivation



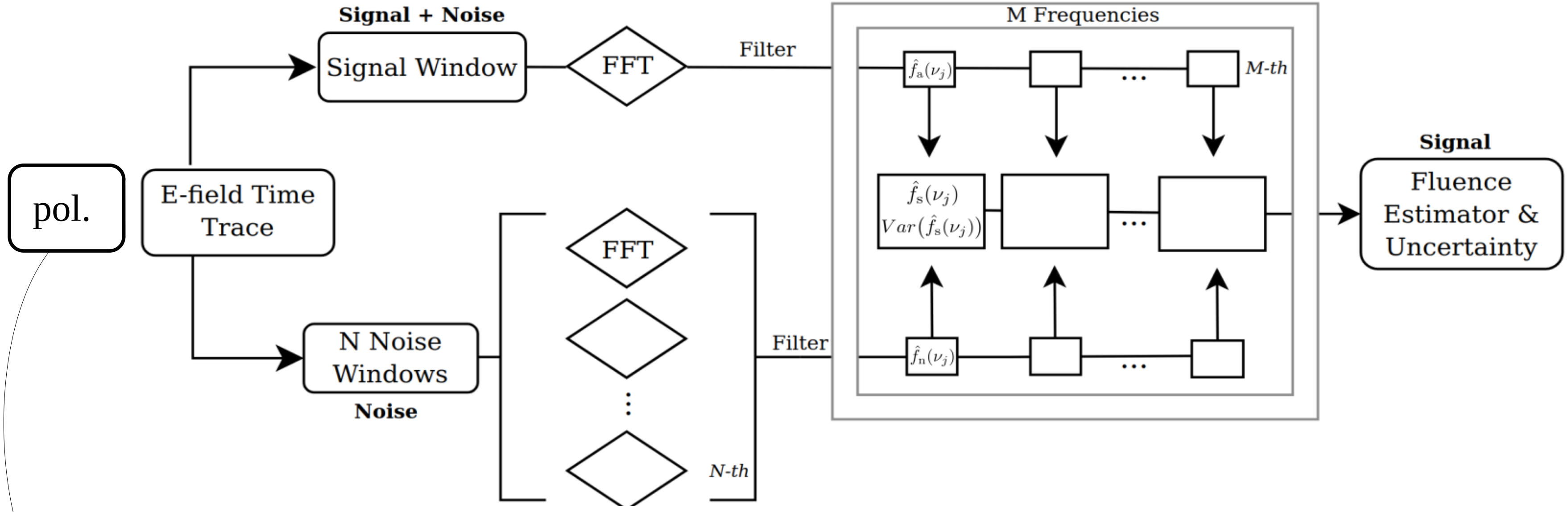
$$\hat{f}_s = \sum_{j=0}^{M-1} \hat{f}_s(\nu_j)$$

$$Var(\hat{f}_s) = \sum_{j=0}^{M-1} Var(\hat{f}_s(\nu_j))$$



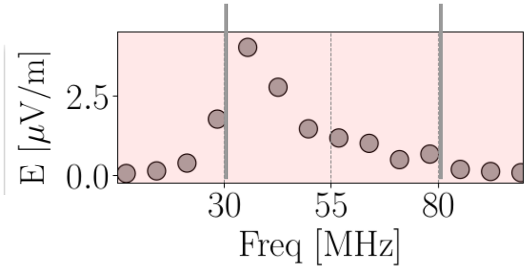


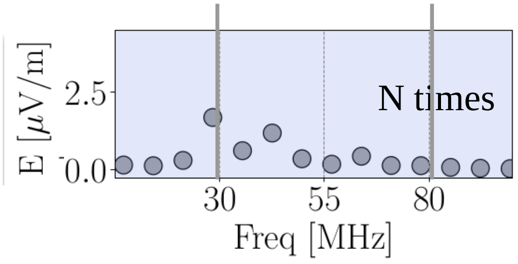
Rice-distribution Based Method: Derivation



$$\hat{f}_{\mathrm{tot}}(\vec{r}) = \sum_{\mathrm{pol}}^{3} \hat{f}_{\mathrm{pol}}(\vec{r})$$

Error propagation

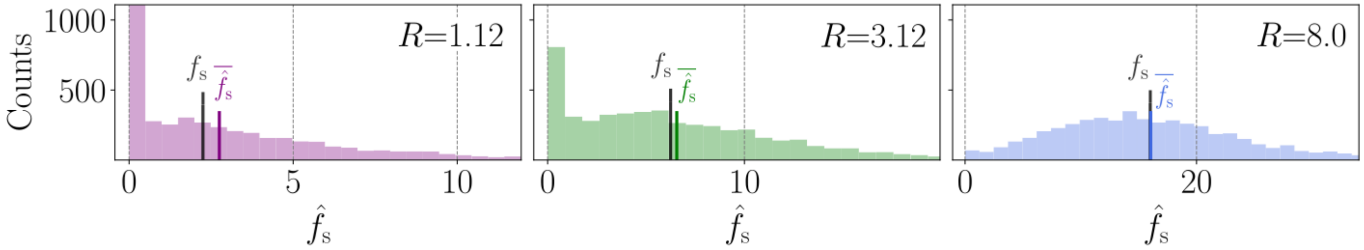




Rice-distribution Based Method: Intrinsic Bias & Coverage

Toy Monte Carlo (using the Rice P.D.F. and the estimator derived)

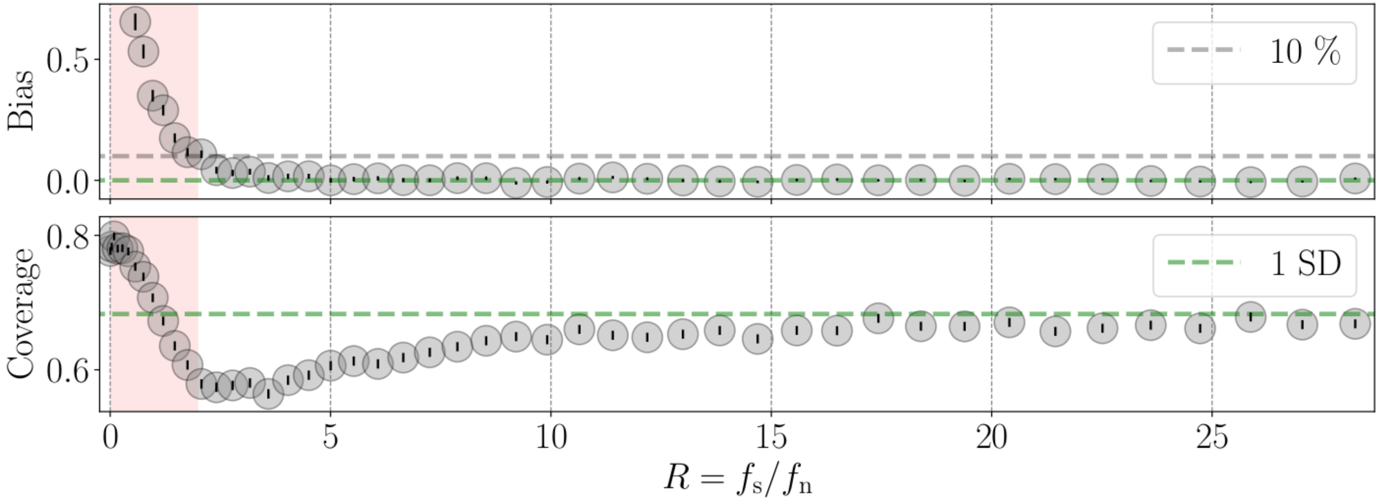
$$\hat{f}_s = \begin{cases} \hat{f}_a - \hat{f}_n \text{ if } \hat{f}_a \geq \hat{f}_n \\ 0 \text{ if } \hat{f}_a < \hat{f}_n \end{cases}$$



$$\psi = \overline{\hat{f}_s}/f_s - 1,$$

$$\sigma_\psi = \sqrt{\sum_{i=0}^{N_{MC}-1} \frac{(\hat{f}_{s,i} - \overline{\hat{f}_s})^2}{N_{MC} - 1}}$$

$$f_s \in [\hat{f}_s - \delta(\hat{f}_s), \hat{f}_s + \delta(\hat{f}_s)]$$



Noise Subtraction Method: Derivation

Reference: C. Glaser's PhD thesis

5.5.2 Uncertainty of the energy fluence

We estimate the uncertainty of the energy fluence by assuming that the measured electric-field amplitude $A(t)$ is the sum of the cosmic-ray radio pulse $S(t)$ and noise $e(t)$. Furthermore, we assume that the noise $e(t)$ is Gaussian distributed with mean $\mu = 0$ and standard deviation $\sigma = \sigma_e$. The energy fluence of A is then given by the equation

$$f(A) = \epsilon_0 c \Delta t \sum_{t_1}^{t_2} A(t_i)^2 = \epsilon_0 c \Delta t \sum_{t_1}^{t_2} [S(t_i) + e(t_i)]^2 = \epsilon_0 c \Delta t \sum_{t_1}^{t_2} \left[S(t_i)^2 + 2S(t_i)e(t_i) + e(t_i)^2\right] \tag{5.16}$$

and the expectation value of $f(A)$ is

$$\begin{aligned} \langle f(A) \rangle =& \epsilon_0 c \Delta t \sum_{t_1}^{t_2} A(t_i)^2 = \epsilon_0 c \Delta t \sum_{t_1}^{t_2} \left[\langle S(t_i)^2 \rangle + 2\langle S(t_i) e(t_i) \rangle + \langle e(t_i)^2 \rangle\right] \\ =& \epsilon_0 c \Delta t \sum_{t_1}^{t_2} \Bigg[\langle S(t_i) \rangle^2 + \underbrace{Var(S(t_i))}_{=0} + 2\langle S(t_i) \rangle \underbrace{\langle e(t_i) \rangle}_{=0} \\ &+ 2\,\underbrace{Cov(S(t_i), e(t_i))}_{=0} + \underbrace{\langle e(t_i) \rangle^2}_{=0} + \underbrace{Var(e(t_i))}_{\sigma_e^2} \Bigg] \end{aligned} \tag{5.17}$$

$$= \epsilon_0 c \Delta t \sum_{t_1}^{t_2} \left[\langle S(t_i) \rangle^2 + \sigma_e^2\right] .$$

Hence, the best estimate of the energy fluence of the radio signal S is indeed

$$f(S) = \epsilon_0 c \Delta t \sum_{t_1}^{t_2} \left[A(t_i)^2 - \sigma_e^2\right] \tag{5.18}$$

as defined in Eq. (5.8) where σ_e^2 is also calculated from the electric-field trace in a part where no signal is present. Following a similar calculation we can estimate the uncertainty of $f(S)$ by computing $\sigma_f^2 = Var(f) = \langle f^2 \rangle - \langle f \rangle^2$. After several lines of calculation it follows that

$$\sigma_f^2 = 4f\, \epsilon_0 c\, \Delta t\, \sigma_e^2 + 2\,(\epsilon_0 c)^2\,(t_2 - t_1)\,\Delta t\, \sigma_e^4 . \tag{5.19}$$